PROMOTING STUDENTS' REFLECTIVE THINKING OF MULTIPLE QUANTIFICATIONS VIA THE MAYAN ACTIVITY

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This research explored principles in designing an instructional intervention to promote students' reflective thinking about multiple quantifications. We suggested the Mayan activity and examined to what extent it promoted students' reflective thinking about the independence of ε from N in the ε-N definition of the limit of a sequence. After the Mayan activity, students recognized the problem caused by describing ε as dependent on N and hence understood the significance of the order between them. Students also developed proper reasoning and tested their hypotheses by using the Mayan stonecutter story. These results indicate that the Mayan activity provides an example logically compatible with describing ε as dependent on N, tractable, and transferable.

Introduction

The aim of this study is to explore design principles to promote students' reflective thinking about the independence of ε from N in the ε-N definition of the limit of a sequence. Although the ε-N definition is a fundamental idea in studying advanced mathematics, many students encounter difficulty in understanding the ε-N definition (e.g., Cornu, 1991; Davis & Vinner, 1986; Edwards, 1997). Students' difficulty is partially caused by the lack of understanding of multiple quantifications in general. Dubinsky and Yiparaki (2000) referred to statements of the form "for all $x \in X$, there exists $y \in Y$ such that $P(x, y)$," where $P(x, y)$ is a statement about variables x and y, as AE statements; EA statements are those of the form "there exists $y \in Y$ such that for all $x \in X$, $P(x, y)$." Students in general tend to perceive EA statements as AE statements. For instance, for a statement "there is a flower that every bug likes," undergraduate students in introductory proof courses tended to interpret it as "for every bug, there is a flower that the bug likes" (Roh, 2011). When an EA statement includes mathematical contents, even more students tend to perceive the EA statement as an AE statement. In Dawkins and Roh's (2011) study, when given an EA statement "there exists a real number x such that for any $\varepsilon > 0$, $|x| < \varepsilon$," seventy-five percent of real analysis students initially perceived the EA statement as the AE statement "for any $\varepsilon > 0$, there exists a real number x such that $|x| < \varepsilon$."

Noticing the order of variables is important in understanding the logical structure of the ε-N definition. However, understanding *why* the order of variables matters is even more important. Literature about the teaching and learning of the ε-N definition (e.g., Burn, 2005; Cory & Giarofalo, 2011; Durand-Guerrier & Arsac, 2005; Roh, 2010a) pointed out the importance of "the dependence rule;" in the AE statement, the value of y may depend on the value of x. By the dependence rule, we may choose the value of N depending on the value of ε when proving the convergence of a sequence using "*for any* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* for all $n > N$, $|a_n - L| < \varepsilon$." On the other hand, Roh (2009) highlighted the importance of "the independence rule;" in both AE and EA statements: the value of x must be independent of the value of y. For instance, when proving the convergence of a sequence, the value of ε *must* be independent of the value of N. Similarly, when disproving the convergence of a sequence using "*there exists* $\varepsilon > 0$ *such that for any* $N \in \mathbb{N}$, there exists $n > N$ such that $|a_n - L| \geq \varepsilon$," the value of ε *must* be independent of the value of N. It is because ε is specified prior to N in these statements. The independence rule seems much harder than the dependence rule for students to grasp in the ε-N definition. When encountering invalid arguments in which ε is selected dependently on N, students experienced difficulty in figuring out what was wrong (Roh, 2009, 2010b; Roh & Lee, in press). Therefore, in order to help students understand the ε-N definition, an instructional intervention is needed to promote students' perception of the independence of ε from N.

This study suggests the Mayan activity as an instructional intervention and gives an account of its effects on students' reflective thinking in the aforementioned context. The Mayan activity consists of three steps: the first is to evaluate an argument which, by selecting ε as dependent on N, leads to the conclusion that a well-known convergent sequence is divergent; the second is to evaluate the Mayan stonecutter story (see Figure 1), which is compatible with, but more tractable than the argument in the first step; and the third is to evaluate arguments, which are described by selecting ε as dependent on N but are more complicated than the argument in the first step.

The Mayan Stonecutter Story

One of the famous Mayan architectural techniques is to build a structure with stones. These stones were ground so smoothly that there was almost no gap between two stones. It was even hard to put a razor blade between them. One day a priest came to a craftsman to request smooth stones.

Craftsman: *No matter how small of a gap you request, I can make stones as flat as you request if you give me some time.*

Priest: *I do not believe you can do it. If I ask you to flatten stones within 0.01 mm, you won't be able to do it.*

Craftsman: *Give me 10 days, and you will receive stones as flat as within 0.01 mm.*

Ten days later, the craftsman made two stones so flat that the gap between them was within 0.01 *mm*. On the 11th day, the priest came to see the stones and argued that,

Priest: *These stones are not flat within 0.001 mm. What I actually need are stones as flat as within 0.001 mm.*

Craftsman: *Okay, if you give me 5 more days, I can make the stones as flat as within 0.001 mm.*

Five days later, the craftsman made the two stones so flat that the gap between them was within 0.001 *mm*. On the 16th day, the priest came to see the stones and argued that,

Priest: *But these stones are not flat within 0.0001 mm and I meant 0.0001 mm. You don't have that kind of skill, do you?*

If the priest keeps arguing this way, is the priest really fair showing that the craftsman does not have the ability to flatten stones within any margin of error?

Figure 1 **The Mayan Stonecutter Story**

In order to evaluate the validity of the priest's argument, it is necessary to examine the relations between the margin of error and the day in the priest's argument and in the craftsman's claim, respectively. On one hand, the margin of error in the craftsman's claim is independently chosen from the day. On the other hand, the priest attempted to argue the negation of the craftsman's claim to reveal the fallacy of the craftsman's claim. However, the priest committed a mistake choosing, actually requesting, a new margin of error *dependently* on the day in his argument even if the margin of error must have been chosen *independently* from the day in the negation of the craftsman's claim. We examined how the Mayan activity leveraged students' reflective thinking about the independence rule not only during the activity but also afterwards. Through this examination, we propose design principles for instructional interventions to promote students' reflective thinking.

Theoretical Perspective: Reflective Thinking

Reflective thinking has long been of interest and broadly studied in the area of the teaching and learning of mathematics. For instance, Dubinsky (1991), Piaget (1985), and Simon (2004) examined the role of reflective thinking in the abstraction of mathematical concepts, focusing on individuals' cognitive development. Cobb, Boufi, McClain, and Whitenack (1997), and Hershkowitz and Schwarz (1999) focused on how social interactions in classrooms influence reflective thinking. Also, mathematics teachers' ability to reflect on their own teaching was explored by Artzt (1999), Lee (2005), and Silverman and Thompson (2008).

In order to build an instructional intervention for the independence rule in the context of convergence, we adopted Dewey's theory of reflective thinking. According to Dewey (1933), when an individual encounters a situation contradictory to his or her knowledge or belief, he or she experiences perplexity or frustration, then resolves it through reflective thinking. Dewey regards reflective thinking to be a continuum of the following three situations: the *pre-reflective* situation, a situation experiencing perplexity, confusion, or doubts; the *post-reflective* situation, a situation in which such perplexity, confusion, or doubts are dispelled; and the *reflective* situation, a transitionary situation from the pre-reflective situation to the post-reflective situation. In addition, Dewey characterizes the reflective situation in terms of five phases, aspects, or functions as follows: suggestions, an intellectualization, hypotheses, reasoning, and testing by (mental) actions. *Suggestions* refer to the function of suggesting merely some primitive idea(s) inhibiting a direct action when one experiences perplexity, confusion, or doubts. *Intellectualization* means the phase that the perplexity or the difficulty is defined or rises to the surface of thought. In the intellectualization phase, one is not merely emotionally annoyed or embarrassed by the difficulty, but figures out what the difficulty is as an intellectual problem. Through intellectualization, the ideas proposed in the suggestion phase can be changed or modified into more appropriate ones to resolve the difficulty. Such enhanced ideas are called *guiding* suggestions or *hypotheses*. Whereas the phases of suggestions or hypotheses examine ideas via observations, the *reasoning* phase is the phase to postulate meaningful results without observations. *Testing by (mental) actions* is the phase to verify or to confirm the hypotheses.

Research Methodology

This research was classified as a one-on-one design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) comprised of a series of teaching sessions. By the exploratory nature of the teaching sessions (Steffe & Thompson, 2000), this research attempted to address claims about how students develop their reflective thinking related to the contexts of the convergence of sequences. The design experiment was iterated four times between the fall semester of 2006 and the spring semester of 2010. The iterative nature of the design experiment allowed us frequent cycles of design issues, (re)design of instructional interventions, prediction of student learning, implementation of the instructional interventions, and analysis of actual student learning.

In this paper, we report two studies. Each study was conducted in an advanced calculus class at a public university in the USA in the fall semester of 2006 (Study 1) and in the spring semester of 2010 (Study 2). The subjects of the studies were undergraduate students who majored in either mathematics or education with a concentration in mathematics. Calculus and introductory proof courses were prerequisite for the advanced calculus course. The first author of this paper served as an instructor in both studies. In Study 1, the advanced calculus class consisted of a series of 50-minute sessions, three times per week for 15 weeks. In Study 2, the advanced calculus class consisted of a series of two regular sessions (75 minutes in length) and a recitation (50 minutes in length, mainly for cooperative proof writing) every week for 15 weeks.

Instruction in the classes mainly followed an inquiry approach, in which students often made and justified conjectures, or evaluated given arguments. In this manner, for the first four weeks, the students explored the real number system and its related properties, and reviewed elementary logic and proof structures. For the subsequent two weeks, the students studied the limit of a sequence and some relevant properties. In particular, they dealt with the ε-N definition of the convergence of sequences, and proofs of the convergence or divergence of given sequences by applying the ε-N definition. Also, discussions in the same manner followed dealing with the definition of Cauchy sequences and their relevant properties.

Throughout both studies, students' activities were video-recorded and were then transcribed. Also, we scanned students' paper-and-pencil work including homework and exams. Dewey's theory of reflective thinking (1933) was employed as a framework for our design experiment, in particular, in the steps of raising design issues, predicting and analyzing student learning, and (re)designing instructional interventions. After abstracting the students' language expressions, gestures, and diagrams from the transcripts of the video recordings, we first coded students' evaluations into one of the following: the pre-reflective situation, reflective situation, or post-reflective situation. Next, we coded the data into one of the aspects of the reflective situation, i.e., suggestions, an intellectualization, hypotheses, reasoning, or testing by (mental) action. Through such a coding process, we analyzed whether the students experienced or resolved perplexity and to what extent reflective thinking was enhanced.

Results from Study 1

Design Issue 1 with Statement 1 and Ben's Argument

In designing an instructional intervention to promote students' reflective thinking about the independence of ε from N, the first design issue centered around how to make students perceive that selecting ε dependently on N causes a problem. Accordingly, we designed Statement 1 and Ben's argument to Statement 1 (see Figure 2).

Statement 1: If a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is a Cauchy sequence, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, .

Ben's argument: Consider $a_n = 1/n$ for any $n \in \mathbb{N}$. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to 0, it is a Cauchy sequence in $\mathbb{R}$. Let for all $N \in \mathbb{N}$. Let . Then $n > N$. But $|a_n - a_{n+1}| = |a_{N+1} - a_{N+2}| = \left|\frac{1}{N+1} - \frac{1}{N+2}\right| = \frac{1}{(N+1)(N+2)} = \varepsilon \geq \varepsilon$. Therefore, Statement 1 is false. □

Figure 2 **Statement 1 and Ben's argument**

We implemented the evaluation of Statement 1 and Ben's argument after six weeks of the class in Study 1. Six students volunteered to participate in the study (Study 1) and worked in two small groups, each of which had three members. In this paper, we use pseudonyms Ellen, Simon, and Susan for the students in Group 1 and Max, Mimi, and Scott for the students in Group 2. In fact, Statement 1 is true, but Ben's argument is an invalid argument in which ε is determined dependently on N. We expected that if students do not recognize the problem of the dependence of ε on N in Ben's argument, they would be perplexed and initiate their reflective thinking to resolve the perplexity.

Reflective thinking about the independence of ε from N in Study 1. All students in Study 1 determined that Statement 1 should be true and as the reason for their determination, they pointed out that $n+1$ can be substituted for *m*. After the students' determination of the truth of Statement 1, the instructor introduced Ben's argument. All students seemed to recognize that Ben's argument was contradictory to Statement 1 and became perplexed, which indicates that they were in the pre-reflective situation. We further analyzed the extent of students' reflective thinking about the independence of ε from *N*.

In Group 1, Susan raised a question whether choosing $\varepsilon = 1/[(N+1)(N+2)]$ might be a problem in Ben's argument because "it [ε] can change […] as […] *N* changes." However, Ellen insisted, and Susan agreed, that ε can be chosen in such a manner. Later, Susan and Ellen accepted Ben's argument, and hence they reversed themselves over their determination of Statement 1. In fact, these two students suggested Statement 1 was false, and considered Ben's argument to be valid. Since they resolved their perplexity in the wrong direction, they could not perform any proper aspects of reflective thinking about the independence of ε from *N*. On the other hand, Simon was perplexed after reading Ben's argument since he believed Statement 1 to be true but failed to find any fallacy in Ben's argument. Simon said "I don't know. I'm stuck" after Susan said "I think it [Ben's argument] makes sense." Simon would not change his determination of the truth of Statement 1, but he failed to find any fallacy in Ben's argument. Hence, he just remained in the pre-reflective situation without performing any other aspects of reflective thinking about the independence of ε from *N* in evaluating Ben's argument.

All students in Group 2 also determined Statement 1 to be true and seemed to recognize that Ben's argument was contradictory to Statement 1. Max and Mimi thought that it was not a problem to determine ε dependently on *N* in Ben's argument, and consequently they accepted Ben's argument to be valid. Only Scott did not agree that ε should depend on *N*. Max and Mimi accepted Ben's argument, saying that they "can't find anything wrong with it." Furthermore, although Scott pointed out that choosing ε as dependent on *N* is a problem in Ben's argument, Max and Mimi had no doubt about the validity of Ben's argument, saying "Why not?" (Max) and "I'm not sure that's a problem" (Mimi). By accepting Ben's argument, they reversed their determination of Statement 1. Since Max and Mimi resolved their perplexity in the wrong direction, they could not perform any proper aspects of reflective thinking about the independence of ε from *N*. On the other hand, Scott was uncomfortable about Ben's argument, saying "I don't like it." He seemed to consider that Ben's argument was not valid, and developed such a primitive idea into a hypothesis that the dependence of ε on *N* in Ben's argument may cause a problem. However, as an intellectualization in the process of deriving the hypothesis, Scott just recalled that "I made that mistake 3 or 4 times on my homework." Also, he could not perform any other proper reasoning aspect of the reflective thinking about the independence of ε from *N*, and consequently he failed to resolve his perplexity.

Results from Study 2

Design Issue 2

We can expect that when two conflicting arguments are suggested, students can recognize that at least one of the arguments is false. However, it is not assured that they will select the true statement between the two conflicting arguments. In Study 1, many students accepted Ben's argument contradictory to Statement 1 even though they determined Statement 1 to be true in the previous step. Although their reasoning was proper in deriving the truth of Statement 1, they reversed themselves about the truth of Statement 1 once they could not find that the fallacy of Ben's argument is induced by describing ε dependently on *N*. Therefore, in order to properly promote students' reflective thinking about the independence of ε from N, a new design issue centered around a way to exclude the possibility that students might regard a faulty argument described by choosing ε dependent on N to be true. Toward this end, we designed the Mayan activity. After five weeks of the class in Study 2, the instructor implemented the Mayan activity as an instructional intervention. Eleven students volunteered to participate, working in three groups, each of which had three or four members. In this paper, we focus on one group of four students: Amy, Elise, John, and Matt (all student names are pseudonyms).

The Mayan activity – Step 1: Evaluation of Sam and Bill's Arguments. In Step 1 of the Mayan activity, the students evaluated a pair of conflicting arguments about the convergence of a well-known sequence $\{1/n\}_{n=1}^{\infty}$ shown in Figure 3:

> ***Sam's argument.*** For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $N\varepsilon > 1$. Then for all $n > N$,
>
> Therefore, $\{1/n\}_{n=1}^{\infty}$ converges to 0. □
>
> ***Bill's argument.*** For all $N \in \mathbb{N}$, choose . Then $\varepsilon > 0$ since $N > 0$. Let .
> Then $n > N$. Also, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} = \frac{1}{N+1} > \frac{1}{N+2} = \varepsilon$. Therefore, $\{1/n\}_{n=1}^{\infty}$ does not converge to 0. □

Figure 3 **The Mayan activity – Step 1: Sam's and Bill's arguments**

It was observed that all students in this group instantly perceived Bill's argument to be invalid. Amy and John were certain of the validity of Sam's argument and recognized that Bill's argument was contradictory to Sam's argument. In determining the validity of Sam's argument, they reflected on their previous experience that they had proved the convergence of $\{1/n\}_{n=1}^{\infty}$. Elise discerned that Bill's argument attempted to negate, but not properly, the definition of the convergence of sequences. However, she was unable to articulate what was amiss with Bill's argument. On the other hand, Matt perceived that a negation was attempted in Bill's argument, and pointed out that the problem with Bill's argument was that "the quantifiers were all negated […] but out of order." In addition, Matt understood that such an incorrect negation, by reversing the order of variables, allows ε to be described dependently on N. Through the conversation with Matt, John agreed that a negation was attempted in Bill's argument. However, John was still questioning "why does the order matter?" even though Matt pointed out that a problem bears on the order of the variables.

Matt: Is the main error the fact that you know ε is dependent on N? […] So if we have to pick one major flaw in logic, would that be it in this one? […]

Elise: […] He [Bill] is trying to negate this one, and he just does it incorrectly. He just moves it around. … He's trying to negate it. … Just that's what I see it … because the negation just made me sit there and go what? […]

Matt: So then how did he [Bill] not correctly negate it? I guess that's part of the question here.

John: Yeah, that's what I kind of don't understand here. Why does the order matter?

Matt pointed out that the logical problem in Bill's argument was to choose ε dependently on N. However, other students in this group did not seem to understand his assertion. In particular, Amy did not participate in the group discussion except arguing that Bill's argument is incorrect. Elise and John perceived that a negation was attempted in Bill's argument, but they did not understand why the reversal of the order between ε and N would be a problem in Bill's argument.

Reflective thinking about the independence of ε from N in Step 1. We could observe that the students in Study 2 immediately noticed that Sam's argument was valid and Bill's argument was contradictory to Sam's argument; thereafter, they suggested Bill's argument was invalid. From that moment, the students' reflective thinking about the independence of ε from N seemed beyond the pre-reflective situation, and was shifted to the reflective situation. On the other hand, after Amy suggested that Bill's argument was wrong, she did not perform any other aspects of the reflective situation. Unlike the case of Amy, Matt intellectualized the problem of Bill's argument, pointing out that a negation was attempted. Further, he suggested a hypothesis that the problem with Bill's argument bore on the order of quantifiers in the attempted negation, which was in fact reversed from that in the correct negation of the ε-N definition. Also, he reasoned that reversing the order of quantifiers in such a way allows ε to be described dependently on N and that Bill's argument could be unfolded due to the dependence of ε on N. Through such reasoning, Matt was able to resolve his perplexity caused by Bill's argument. Also, Elise and John explored intellectually the problem of Bill's argument, taking note that a negation was attempted. However, when Matt asked, "How did he [Bill] not correctly negate it? I guess that's part of the question here," John responded, "That's what I kind of don't understand here," and asked back to the group, "Why does the order matter?" Unlike the case of Matt, Elise and John failed to develop proper hypotheses and reasoning with regard to the independence of ε from N. Consequently, Amy, Elise, and John could not resolve their perplexity, which was caused by Bill's argument.

Design Issue 3

Results from Step 1 of the Mayan activity indicate that the evaluation step of Sam's and Bill's arguments provides a way to exclude the possibility that students might accept a faulty argument described by selecting ε dependently on N. Considering students' change in their determination of the validity of Statement 1 in Study 1, the design issue raised from Study 1 was successfully resolved through Step 1 of the Mayan activity. Still, only the task of Step 1 of the Mayan activity did not enable students, except in the case of Matt, to resolve their perplexity caused by Bill's argument. In order to resolve students' perplexity, a new design issue centered around a way to promote proper hypotheses and reasoning, hence eventually to develop students' reflective thinking about the independence of ε from N. Thus, we designed and implemented the Mayan stonecutter story (see Figure 1) in Step 2. The students presented their individual evaluation and debated whether the priest was fair or not in group discussion.

The Mayan Activity – Step 2: The Priest's Argument in the Stonecutter Story. During group discussion, the students immediately insisted that the priest was unfair. However, in explaining why the priest was unfair, they presented different bases to each other. John insisted that the priest did not give the craftsman enough time to flatten stones as requested, and hence the priest was unfair. Amy misunderstood the craftsman's claim to be "if the craftsman has some time, then he can make *some* very flat stone." In fact, on the basis that the craftsman made stones flat within *some* margin of error, she decided that the craftsman had already fulfilled his requirement. For this reason, Amy concluded that the priest's argument was not fair. Elise paid attention to the difference between the logical structures of the craftsman's claim and the priest's argument. Pointing out that the craftsman's claim is structured by "for all, there exists" whereas the priest switched it into "there exists, for all," she concluded that the priest was not fair to the craftsman. It was noted that John, Amy, and Elise compared the priest's argument with the craftsman's claim to explain the priest's fallacy. On the other hand, Matt tried to compare a negation of the craftsman's claim with the priest's argument in his attempt to explain the priest's fallacy. Matt seemed to consider that in order to show the unfairness of the priest, it is sufficient to prove that the priest's argument is equal to the negation of the craftsman's claim. He first represented the craftsman's claim as a quantified one in such a way that "for all distances, there exists a time for which the craftsman can get it done." Next, he attempted to describe the negation of his representation of the craftsman's claim and to compare it with the quantified one of the priest's argument. However, Matt fell into a logical mistake of stating the negation of the craftsman's claim to be "in a certain time, there exists a distance that the craftsman can't get to."

Through the subsequent group discussion, Matt and Elise focused on the logical structure of the priest's argument compared to that of the craftsman's claim. When Matt tried to figure out the logical meaning of the priest's argument, Elise asked him to repeat his logical interpretation of the craftsman's claim and suggested a negation of his logical interpretation in terms of a quantified statement.

Elise: What was your original one? The original like regular quantifiers, not negated? Or, what was the statement you had?

Matt: It was for all – um distances or – you know – gaps, there exists an amount of time for which the craftsman can get that distance.

Elise: Yeah, […] I think […] your negation is that there exists a distance between them such that for all time you give them – for all there – you can't make it or that.

Matt: Yeah. […] Well, and maybe he [the priest] is flipping them. He is saying for all time, there is – the existence of – a distance that you can't get to.

Elise: Yeah.

Matt: Well, flipping those is a big deal! (laughs)

Elise: (laughs)

Matt: Because if you're saying – you know – there exists a distance that no matter how much time he has, he can't get to. That's way different from saying – you know – for any times, there – you know, at this time, there exists – you know – one thing that doesn't work. Well, that doesn't tell us a whole lot about the negation of the original statement – you know –

Elise: Yeah.

Matt: but tells us what happens at that one time. And then – but then the distance is dependent on whatever time. If we're looking at any time, then it [distance] becomes dependent on that time. That distance doesn't work, so at a different time, other distance doesn't work.

Elise: Yeah.

Matt: So that becomes – it seems a big deal at that point!

Elise could suggest the correct negation of the craftsman's claim. Taking Elise's suggestion into account, Matt could recognize that the order of quantified variables, distance and time, in the priest's argument is reversed from that in the negation of the craftsman's claim. Furthermore, Elise and Matt reasoned that the reversed order of quantified variables permitted the dependence of 'distance' on 'time,' hence caused the logical problem in the priest's argument. Later, Matt pointed out that the order of quantifiers was flipped in the priest's argument, and hence the priest's argument did not give a disproof of the craftsman's claim. After listening to Matt's explanation, John recognized that he had not taken into account the dependent relationship between the variables, 'time' and 'distance.' He tried to articulate the priest's argument in terms of the distance requested and the time allowed to the craftsman. In particular, John pointed out that "some time" means the time the craftsman suggested for the previously requested distance, but the priest insisted that the craftsman did not make the newly requested distance at that time. Elise suggested further a logical approach to the problem of the priest's argument. She pointed out that for the priest to negate the craftsman's claim, the priest would have to show that "there exists a distance that no matter how much time the craftsman has, he [the craftsman] cannot make that distance," but instead the priest treated the situation as "no matter how much time the craftsman has, there was a gap that he hadn't reached yet." Other students in the group agreed upon Elise's assertion. Later, the students even compared Bill's argument with the priest's argument in terms of the logical structure. In particular, they responded that the two arguments contain a common logical problem, reversing the order of quantifiers from the desired one to draw each conclusion.

Reflective thinking about the independence of ε from N in Step 2. In Step 2, the students suggested immediately the unfairness of the priest. Also, they perceived that the priest attempted to disprove the craftsman's claim. In particular, Elise and Matt intellectualized that in order to disprove the craftsman's claim, the priest must have proven the negation of the craftsman's claim. They then compared the negation of the craftsman's claim with the priest's argument in terms of quantified statements. Afterwards, they came to recognize that the logical order between the margin of error and time in the priest's argument was reversed from that in the negation of the craftsman's claim. Elise and Matt hypothesized that the reversal of the variables in the priest's argument from that in the craftsman's claim was the crucial problem. They further reasoned that in attempting to disprove the craftsman's claim, the reversed order of the quantified variables in the priest's argument permitted the dependence of 'distance' on 'time', hence the priest's argument was irrelevant to the negation of the craftsman's claim. Through the group discussion, John revised his primitive idea and reasoned that the problem of the priest's argument was not to give sufficient time to the craftsman. In particular, John pointed out that the priest just gave the craftsman the time corresponding to the previously requested distance, but requested a new distance without giving the time corresponding to the newly requested one. Like this, Elise, John, and Matt resolved their perplexity due to the priest's argument by developing proper hypotheses and through proper reasoning. On the other hand, Amy's suggestion of the priests' unfairness was based on her misunderstanding of the craftsman's claim as "if the craftsman has some time, then he can make *some* very flat stone." In her misunderstanding of the craftsman's claim, Amy improperly intellectualized the problem of the priest's argument by regarding *any* margin of error as *some* margin of error in the craftsman's claim. Nonetheless, considering her lack of participation in Step 1, Amy's reflective thinking was substantially enhanced through the activity with the Mayan stonecutter story in Step 2.

By comparing the priest's argument with Bill's argument, the students came to hypothesize the problem with Bill's argument properly, that is, the order between ε and N in Bill's argument is reversed from that in the negation of the ε-N definition. The students also reasoned that such a reversal of the order between the variables ε and N leads to an erroneous conclusion from Bill's argument. Consequently, reflecting on their thinking process about the priest's argument, the students could resolve their perplexity caused by the problem with Bill's argument.

Design Issue 4

The results from Step 2 indicate that through the activity with the Mayan stonecutter story, students could enhance their reflective thinking about the independence of ε from N and hence resolve their perplexity due to the priest's argument in the story. In fact, they could promote proper hypotheses and reasoning through the activity with the Mayan stonecutter story; consequently, the design issue newly raised from Step 1 seemed to be achieved in Step 2.
We now consider a fourth design issue: how students utilize their reflective thinking

when evaluating different arguments with the same fallacy of determining ε dependently on N as in the priest's argument.

The Mayan Activity – Step 3: Evaluation of Jane's Argument. Immediately after Step 2 of the Mayan activity, the class dealt with some concepts and results relative to sequences including Cauchy sequences. In the subsequent week, we asked the students to evaluate Jane's argument for Question 6 (see Figure 4). Jane's argument was individually evaluated by a paper-and-pencil test and then by follow-up interviews outside of class, where the interviwer was the first author of this paper.

Question 6. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = 1/2^n$ for any $N \in \mathbb{N}$. Jane was asked to prove or disprove that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to 0. Evaluate if Jane's argument below can be regarded as a legitimate proof. Explain how you can tell.

Jane's argument. For all $N \in \mathbb{N}$, choose $\varepsilon = \dfrac{1}{2^{N+2}}$. Let   . Then $n > N$. Also $|a_n - 0| = \left|\dfrac{1}{2^n} - 0\right| = \dfrac{1}{2^n} = \dfrac{1}{2^{N+1}} > \dfrac{1}{2^{N+2}} = \varepsilon.$ Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge to 0. □

Figure 4 **The Mayan activity – Step 3: Question 6 and Jane's argument**

Jane's argument draws an erroneous conclusion by selecting ε depending on N as in Bill's argument in Step 1 of the Mayan activity. Bill's argument, the priest's argument, and Jane's argument all include a logically identical error in the sense that ε (the margin of error) is selected dependently on N (the day). Comparing students' evaluation of Jane's argument with their evaluation of Bill's argument and the priest's argument, we explored how students' reflective thinking about the independence of ε from N has been promoted after the implementation of the Mayan stonecutter story.

In the paper-and-pencil test, all students determined Jane's argument to be invalid. In the follow-up interview, Amy responded that Jane's argument attemped to prove the negation of the definition of convergence. Amy was also aware that the order between variables, ε and N, in Jane's argument was reversed from that in the negation of the definition of convergence. She pointed out that the problem in Jane's argument was to select ε dependently on N, although ε must have been chosen independently from N.

Amy: She [Jane] wants to prove $\{a_n\}_{n=1}^{\infty}$ is not convergent to 0. So she uses the negation – negation of the definition of convergence. She had to do there exists ε – which is greater than 0 – for all $N \in \mathbb{N}$, there exists n – which is greater than N – such that […] $|a_n - L| \geq \varepsilon$. But what she actually did was that she changed the order of the negation of definition. […] she [Jane] has to choose ε first and that condition [$|a_n - 0| \geq \varepsilon$] is satisfied for all $N \in \mathbb{N}$. But in here she – she gave the condition of the N first and then choose the ε. So – um now, the ε is dependent on the N. But actually she had to choose ε first independently from the N. She had to pick for ε whatever the N is in the natural numbers. But what she actually did is she gave the condition first about the N, and picked for the ε regarding to the N. So, we have the ε which is dependent on the N.

John compared Jane's argument with the negation of the definition of convergence. He noted that unlike the negation of the ε-N definition, Jane's argument selected ε dependently on N in such a way that $\varepsilon = 1/2^{N+2}$. For this reason, John regarded Jane's argument not to be equivalent to the negation of the ε-N definition.

John: The main thing is to prove that, you know, something is convergent. We need to prove that for all $\varepsilon > 0$, there exists N ↗ ☠ such that for all $n > N$, um $|a_n - L| < \varepsilon$. […] Now, she is trying to, you know, to negate this statement here, but she didn't. The correct negation of that is saying that there exists an $\varepsilon > 0$ where for all N ↗ ☠, there exists an $n > N$ such that $a_n - 0 \geq \varepsilon$, all right? This is really her intent. But instead she proved for all N ↗ ☠ N∈N, there exists an $\varepsilon > 0$. These two statements are not equivalent statements.

Roh: Why not?

John: […] It's all based on the dependence of it. So in this— in Jane's argument, you know – she's putting – making ε – you know – dependent on – dependent on N.

Roh: How can you tell?

John: Because she's setting $\varepsilon = 1/2^{N+2}$, so ε is based off of – you know – the N. But in this situation here, ε can be anything. […] ε isn't dependent on anything in particular. […]

Elise and Matt believed that Jane's argument attempted to prove the negation of the convergence but was not properly developed in negating the ε-N definition. Elise pointed out that considering the order of variables, ε depends on N in Jane's argument because ε followed N. For this reason, Elise decided that such a reversal of the order between ε and N caused the problem with Jane's argument. On the other hand, Matt noted that ε in Jane's argument was chosen dependently on N even though Jane's argument must have started off by choosing an ε before N. Consequently, he reasoned that the problem with Jane's argument followed from such a dependence of ε on N. Recalling the priest's argument of the Mayan stonecutter story and comparing it with Jane's argument, Elise pointed out that the order between quantifiers in the priest's argument was reversed in the same way as in Jane's argument. For this reason, she regarded the logical problem with Jane's argument to be the same as that of the priest's argument. Matt also compared Jane's argument with the priest's argument. In his interview, he tried to construe Jane's argument in terms of components in the priest's argument. Identifying ε and N in Jane's argument to the gap in the stones and the time, respectively, Matt pointed out that Jane must have shown the existence of the gap that cannot be achieved no matter how much time had passed; however, Jane just gave a gap that depends on time. Considering his account, we can find that Matt regarded the logical structure of Jane's argument to be the same as that of the priest's argument.

Reflective thinking about the independence of ε from N in Step 3. In Step 3, Amy intellectualized that in Jane's argument, Jane attempted to negate the ε-N definition, and Amy gave a hypothesis that the order of variables ε and N is reversed in the argument. From this hypothesis, Amy reasoned that the reversal of the order of variables ε and N allowed the dependence of ε on N. John also intellectualized that Jane's argument attempted to negate the ε-N definition. Furthermore, he pointed out the dependence of ε on N as a reason why Jane's argument was incorrect as the negation of the ε-N definition. Like Amy and John, Elise and Matt also intellectualized that Jane considered the negation of the ε-N definition. They hypothesized that the order of variables ε and N was reversed by choosing ε after N. From this hypothesis, they reasoned that the dependence of ε on N could be allowed from the reversed order between ε and N. In particular, Elise and Matt could be confident of their reasoning by testing (by mental action) that to reverse the order of the variables in Jane's argument was the same logical fallacy committed in the priest's argument. Consequently, the students all could resolve their perplexity from Jane's argument.

One of the remarkable results from Step 3 was that the students were recalling the

Mayan stonecutter story when evaluating Jane's argument. It should be noted that in Step 3 the interviewer did not guide the students to recall the Mayan stonecutter story, and did not even mention it. Nevertheless, the students compared the logical structures of Jane's argument with that of the priest's argument in the Mayan stonecutter story. Also, through the comparison, students pointed out that describing ε dependently on N is a problem in Jane's argument, and reasoned that such a problem draws an erroneous conclusion as in the priest's argument. By reflecting on the Mayan stonecutter story in their evaluation of Jane's argument, the students could be confident of their evaluation.

Discussions

In this study, we designed an instructional intervention, the Mayan activity, with the intention of enhancing students' reflective thinking about the independence of ε from N. The Mayan activity includes two pivotal instructional components: first, the Mayan activity has a means to cause students' perplexity related to the independence of ε from N. To be precise, it makes students experience perplexity through evaluating some arguments containing erroneous conclusions by describing ε dependently on N. Second, the Mayan activity has a device that enables students to experience first-hand the meaning of the independence of ε from N. In fact, it introduces the Mayan stonecutter story from which students concretely realize the problem of describing ε dependently on N. In the following, we will discuss in detail how the Mayan activity plays a role in enhancing students' reflective thinking about the independence of ε from N.

Necessity of a Convincing Argument

The Mayan activity asks in Step 1 to evaluate Bill's argument, which draws an erroneous conclusion that the sequence $\{1/n\}_{n=1}^{\infty}$ does not converge to 0, by describing ε dependently on N. The rationale for adopting Bill's argument is that almost all students in advanced calculus are confident that the sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0, hence they have no doubt that the conclusion of Bill's argument is not true. The necessity of such an argument was raised from Study 1. In Study 1, Ben's argument was suggested to students instead of Bill's argument. Similar to Bill's argument, Ben's argument was also developed by describing ε dependently on N and drew an erroneous conclusion that Statement 1 was false. However, students' response in evaluating Ben's argument in Study 1 showed a different aspect from that of students in evaluating Bill's argument in Study 2. Just before evaluating Ben's argument in Study 1, students concluded that Statement 1 was true through proper reasoning. However, once failing to figure out the problem in Ben's argument, they changed their determination of the truth of Statement 1. This change of the students' decision about Statement 1 seemed to occur because they failed to perceive that describing ε dependently on N causes a problem in Ben's argument and they could not be confident of their decision about Statement 1. Therefore, in order that students perceive the problem of describing ε dependently on N and develop reflective thinking about the independence of ε from N, it was necessary to suggest some statements, instead of Statement 1, of which students could be confident. For this reason, arguments about the convergence of the sequence $\{1/n\}_{n=1}^{\infty}$ were given to students in Step 1 of the Mayan activity.

On the other hand, when being asked to evaluate two arguments that conflict with each other, students will enter the pre-reflective situation as long as they notice such a conflict. However, whether or not students progress in their reflective thinking in the direction of resolving their perplexity depends on their confidence about the truth of given arguments. Students' activities of evaluating Ben's argument in Study 1 and Bill's argument in Study 2 are a good example: in the former, students were not confident of the truth of Statement

1; hence, they could not develop properly their reflective thinking, whereas in the latter case they could. Therefore, when designing an instructional intervention to enhance students' reflective thinking, it is necessary to consider statements that the students can be confident of their truth, such as the convergence of the sequence $\{1/n\}_{n=1}^{\infty}$ in Step 1 of the Mayan activity. Through such an instructional intervention, we can help students properly develop their reflective thinking.

Necessity of Logically Compatible, Tractable, and Transferable Examples

The Mayan activity, in Step 2, provides the Mayan stonecutter story and asks students to evaluate the priest's argument in the story. The Mayan stonecutter story consists of the craftsman's claim and the priest's argument. Such a structure of the story is *logically compatible* with that of Step 1 consisting of Sam's and Bill's arguments, being contradictory to each other (see Table 1). The priest's argument also draws a logically erroneous conclusion in a similar way by describing ε dependently on N in Bill's argument. The priest's argument just differs from Bill's argument in the sense that it uses gaps between stones and days instead of the variables ε and N.

Sam's argument	*Craftsman's claim*
For any $\varepsilon > 0$	No matter how small of a gap (you request)
there exists $N \in \mathbb{N}$ (such that)	some time (I can make stones as flat as you request)
Bill's argument	*Priest's argument*
For all $N \in \mathbb{N}$	Ten days later… five more days later (this means any time)
choose (exist) $\varepsilon = 1/(N+2)$ (such that $1/(N+1) > \varepsilon$)	the stones that craftsman made are not flat within 0.001 *mm*… not flat within 0.0001 *mm* (this means the existence of some gap)

Table 8 **Logical compatibility between statements in the Mayan activity**

The priest's argument is *logically compatible* with but is more *tractable* than Bill's argument so that students easily understand the logical structure and perceive the logical fallacy in the argument. The necessity of such a tractable example was shown by results from Step 1 of the Mayan activity. In Step 1, the students could perceive that Bill's argument induced an erroneous conclusion; nonetheless, they encountered a difficulty in explaining the reason why such an erroneous conclusion could be derived. Their response to Bill's argument is due to their lack of experience with problematic situations in which an argument described by choosing ε dependently on N implies an illogical conclusion. Therefore, in order to enable students to compensate for their lack of experience, hence to overcome their difficulty, it is necessary to force them to perceive that an argument with ε selected dependently on N leads to an unreasonable conclusion, and to confirm that the illogicality of the argument comes from the dependence of ε on N. Thus, the Mayan stonecutter story was designed to provide a tractable situation related to the problem of the dependence of ε on N.

Furthermore, the stonecutter story is a *transferable* example in the sense that students can properly link the variables (gaps between stones and days) in the priest's argument to the variables (ε and N) in Bill's argument. NCTM (2000) and NRC (2001) suggest that mathematics teachers use concrete or colloquial examples to help students understand an abstract mathematical statement. A reason to use such examples is because these examples are logically compatible with the abstract statement and are more understandable. However, for students to properly understand the meaning of the abstract statement through given examples, the examples must be designed to be *transferable*, that is, students by themselves can make a correspondence between key components in the abstract statement and those in the examples, and can perceive the problem of the abstract statement by projecting the problem of the examples onto the abstract statement. In line with this viewpoint, the Mayan stonecutter story is designed to be *transferable*. In fact, the students in Step 2 matched the gaps between the stones and days to ε and N, respectively. Also the students perceived that the problem resulting from the dependence of ε on N in Bill's argument is the same as the logical problem in the priest's argument.

Design Principles

Based on the results from Study 1 and Study 2, we formulated two design principles as testable predictions for a new instructional intervention aimed at enhancing reflective thinking:

(1) *Conflicting but convincing arguments*: Students will notice when two arguments given in the task are in conflict and will be confident of the truth or falsehood of the arguments.

(2) *Logically compatible, tractable, and transferable examples*: The logical structures of the examples are compatible with those of the arguments in the first design principle. Also, the examples are tractable in the sense that students will recognize a logical fallacy and reason that erroneous conclusions might be drawn from the fallacy. Furthermore, the students themselves will identify key logical components in the examples with those in the original arguments and reflect the logical problem of the original argument onto that of the examples.

This paper reports the Mayan activity as an example of a successful instructional intervention to promote students' reflective thinking about the independence of ε from N. As in the case of the Mayan activity, an instructional intervention needs to comprise two steps: the first step is to cause perplexity, confusion, or doubt accompanied with confidence of the truth or falsehood of the given arguments, and the succeeding step is to provide an example, which is not only logically compatible and tractable but also transferable. An example designed in such a way can help students understand the meaning of the logical structure in the targeted context, and retain their understanding. Consequently, the students can utilize their understanding to comprehend the logical meaning of different arguments, which have the same logical structure as the example. Therefore, an instructional intervention designed to enhance reflective thinking should deploy conflicting but convincing arguments and develop examples to be logically compatible, tractable, and transferable.

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