MATHEMATICAL KNOWLEDGE FOR TEACHING: EXEMPLARY HIGH SCHOOL TEACHERS' VIEWS

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Eleven exemplary high school mathematics teachers were interviewed to investigate their views on mathematical knowledge for teaching. Teachers took part in a one-hour interview and discussed a written lesson plan. Teachers believed the following aspects of mathematical knowledge for teaching to be important: (a) building mathematical ideas using prior mathematics, (b) a range of examples that illustrate a mathematical concept, (c) appropriate applications of a concept, (d) connections between mathematical ideas within and beyond the high school curriculum, and (e) various approaches to problem solving. Teachers also discussed the development of their mathematical knowledge for teaching, which they believed came from their teaching experience and personal experiences in addition to coursework.

Key words: Mathematical knowledge for teaching, high school mathematics teachers, mathematics teacher education

Introduction

Researchers and practitioners generally agree that a robust knowledge of mathematics is needed in order to teach mathematics successfully. Some researchers have used the term *mathematical knowledge for teaching* (MKT) to describe the specific knowledge that mathematics teachers require (e.g., Ball, Thames, & Phelps, 2008; Silverman & Thompson, 2008). Although there has been a great deal of research on MKT, most literature has focused on elementary or middle school teachers (e.g., Ball, et al., 2008; Hill, Ball, & Schilling, 2008; Ma, 1999). Less research has been devoted specifically to secondary teachers. Hence, the mathematical knowledge that secondary teachers require in order to teach effectively remains an open question.

Much of the research on MKT has investigated to what extent teachers have aspects of MKT that researchers believe to be important (e.g., Hill, et al., 2008; Kahn, Cooper, & Bethea, 2003; Kiorala, Davis, & Johnson, 2008). Teachers' voices about what they believe are important components of MKT are largely absent from the literature.

Although any secondary math teacher could give insight into MKT that is important to their teaching, exemplary teachers may be especially reflective and hence informative in this regard (Collinson, 1994). In addition, since exemplary teachers are quite successful, they likely have sufficient MKT to teach effectively. Hence, studying exemplary teachers' MKT may help to articulate the necessary components of MKT for effective math teaching.

How teachers develop MKT is also an open question. Many secondary math teacher preparation programs require extensive work in pure mathematics. For example, in New Jersey, secondary math teachers must have 30 credits of undergraduate mathematics, 12 at the junior level or above (State of New Jersey Department of Education [NJDOE], 2010). However, it is unclear whether taking these courses helps to develop MKT (Moreira & David, 2008; Zazkis & Leikin, 2010). Other researchers have argued that MKT develops through teaching itself (e.g., Leikin & Zazkis, 2010). Exemplary teachers may also be particularly aware of the ways in which their MKT developed (Collinson, 1994). The purpose of this study is to explore exemplary teachers' views on the MKT that they believe to be important to their practice and the ways in which they believe this MKT developed.

This study aims to investigate the *mathematical* knowledge used in teaching rather than the *pedagogical* knowledge specific to mathematics. That is, this study looks at MKT with a mathematical lens, seeking subject-matter components of secondary teachers' MKT. As a result, for the purposes of this paper, the term *MKT* refers to this mathematical point of view, unless otherwise specified. However, many researchers agree that MKT is a type of specialized knowledge that combines both subject-matter knowledge and pedagogical knowledge (e.g., Ball, et al., 2008; Leikin & Zazkis, 2010). Hence, viewing MKT from a mathematical perspective does not eliminate pedagogical aspects, and elements of pedagogy are evident both in the literature reviewed for this study as well as the results of this study.

Related Literature

Mathematical knowledge for teaching. A great deal of recent research devoted to mathematics teacher knowledge has emerged in the last 25 years, but only a portion of this research has helped to define the subject-specific components of MKT. Highlights of such research will be described below.

One approach to identifying subject-matter components of MKT has been to synthesize prior research. Kennedy (1998) systematically searched reform documents and research literature to determine the mathematics teachers need to know to teach well. She found that mathematics teaching requires a conceptual understanding of subject matter, which includes (a) understanding the relevant size of numbers, (b) recognizing the central ideas within the subject, (c) understanding the relationships between central ideas, (d) an elaborated and detailed knowledge of the subject, and (e) a profound mathematical reasoning ability.

A second approach used by researchers has been to build frameworks based on prior research and then illustrate how these frameworks can be applied to study teacher knowledge. For example, Even (1990) synthesized related research to develop a framework for evaluating teachers' knowledge of a particular concept. This framework proposed seven components of teachers' knowledge of a particular concept: (a) essential features of the concept, (b) different representations of the concept, (c) alternate ways of approaching the concept, (d) unique characteristics of the concept, (e) a basic repertiore of examples illustrating the concept, (f) procedural and conceptual understanding of a concept, and (g) the nature of mathematics as a discipline. Even then surveyed 162 preservice secondary teachers to investigate their knowledge of the concept of *function.* Follow-up interviews were conducted with 10 of these teachers, and Even's framework was applied to analyze preservice teachers' understanding. A similar approach was used by Chinnappan and Lawson (2005) in the domain of geometry.

A third approach has been to use grounded theory to identify subject-specific components of MKT. Ma's (1999) seminal work described features of the elementary mathematics knowledge of successful Chinese math teachers. These included (a) providing rationales for algorithms, (b) justifying explanations with symbolic derivations, (c) using multiple approaches to computational procedures, and (d) recognizing relationships between mathematical ideas. At the secondary level, Zazkis and Leikin (2010) surveyed 52 math teachers about their use of advanced mathematical knowledge in their teaching. Teachers claimed that because of their advanced mathematical knowledge, they were able to, among other things, (a) make connections within and beyond the high school curriculum, (b) provide alternative solutions problems and alternative representations of mathematical ideas, (c) provide applications of mathematics outside of the classroom, (d) consider why mathematical ideas are true, and (e) use problem-solving skills.

More empirical research that aims to identify the subject-specific components of MKT and provide clear examples of these components is sorely needed, especially at the secondary level. This study will qualitatively explore the aspects of MKT that successful teachers have in order to add to the research base on MKT.

The development of mathematical knowledge for teaching. In order to improve teachers' MKT, it is necessary to understand how this MKT develops. Some research has aimed to describe this development.

As part of her 1999 study, Ma investigated when and how a teacher may develop their MKT. She interviewed Chinese teachers whom she considered to have strong MKT, exploring how these teachers believed they acquired this MKT. Teachers indicated that they developed MKT by intensely studying teaching materials, working with colleagues, working with students, and solving mathematical problems in several ways. Potari, Zachariades, Christou, Kyriazis, and Pitta-Pantazi (2007) observed and interviewed nine calculus teachers to investigate their MKT as well as their beliefs on how their knowledge developed. These teachers generally agreed that their mathematics coursework was not helpful to their development of MKT, but they did find integrated courses, which focused on both mathematics and students' thinking in mathematics, to be helpful.

While both of the studies mentioned here provide insight into the development of MKT, more research is needed in this area. It is unclear how the Chinese teachers' development in Ma's (1999) study might extend to secondary teachers in the United States or if the beliefs discussed in Potari, et al.'s (2007) study hold true for exemplary teachers. The present study will contribute to this research by seeking exemplary secondary teachers' beliefs regarding the development of their MKT.

Research Questions

1. What subject matter components of MKT do exemplary high school teachers believe are important for their practice?
2. When and how do these teachers believe that their MKT developed?

Methods

Participants

Selection process. Participants for this study were drawn from 26 teachers who taught high school (9th through 12th grade) mathematics and had been recognized for their teaching in New Jersey between 1999 and 2010 in at least one of three ways[1]:

1. Ten teachers were state and/ or national finalists for the Presidential Award for Mathematics and Science Teaching (PAMST; National Science Foundation, 2009).
2. Eleven teachers were named County Teacher of the Year (CTOY; NJDOE, 2006).
3. Nine teachers were National Board Certified Teachers (NBCTs; National Board for Professional Teaching Standards, 2010) in adolescence and early adulthood mathematics.

Teachers' knowledge of mathematics and mathematics pedagogy are significant criteria for the three awards listed above.

Six of these 26 teachers could not be located because they had either retired or changed employers since they received their award. As a result, 20 teachers were invited to participate in this study by email or phone. Two declined to participate, and seven did not respond to repeated invitations. The remaining teachers agreed to participate.

Participant characteristics. Eleven teachers agreed to participate in this study: four received the PAMST, three were CTOTY, and seven were NBCTs. (Note that some teachers met more than one criterion. The specific awards for each teacher are not listed in order to maintain confidentiality.) Participants were eight females and three males with teaching experience ranging from 10 to 32 years. The median number of years of experience was 17. All but one of the participants had obtained Master's degrees. One participant was a Doctor of Education, and one participant was a candidate for Doctor of Education.

At the time of the interview, two participants taught at private schools, and eight participants taught at public schools. Teachers were employed by a wide range of districts in terms of socioeconomic status[2]. The New Jersey Monthly magazine 2010 rankings[3] of the public high schools where participants taught ranged from the top 10% in the state to the bottom 20% in the state, with the mean rank being 166 out of 322, and the median ranking being 174 out of 316 (New Jersey Monthly, 2010).

Procedure

Two sources of data were collected to answer the research questions: lesson plans and interviews. Data was collected March through June of 2010.

Lesson plans. Each participant was asked to submit one lesson plan from a high school course (i.e., not Calculus or AP Statistics[4]). Lesson plans were obtained anywhere from one day to one two weeks before the scheduled interviews. Courses in which lessons were taught included Algebra I (one teacher), Geometry (three teachers), Algebra II (five teachers), Precalculus (one teacher), and International Baccalaureate Math (one teacher).

[1] Note that some teachers met more than one criterion.

[2] New Jersey public school districts are assigned district factor groups (DFGs) as approximate measures socioeconomic status. Groups range from A to J: A indicates low socioeconomic status, and J indicates high socioeconomic status. Public school teachers in this sample were employed by districts with DFGs ranging from B to I at the time of the study, with at least one teacher representing every group in between (NJDOE, 2004).

[3] New Jersey Monthly magazine ranks public high schools every two years according to factors such as average class size, dropout rate, percentage of teachers with advanced degrees, SAT scores, state test scores, and so on.

[4] Examples from calculus and AP statistics naturally came out in some interview discussions.

Prior to interviews, mathematical or pedagogical points of the lesson that needed further explanation as well as points of the lesson that had the potential to yield valuable information about teachers' use of their content knowledge were noted, and interview questions were added according to these points. Lesson plans were used during the interviews in order to help prompt concrete examples of how MKT was used in practice.

Interviews. Interviews were semi-structured and lasted approximately one hour. All interviews were audio-recorded, and detailed written notes were taken to record teachers' responses and researcher observations.

The interview protocol included general questions about the teacher's background in math education, both general and specific questions about the lesson plan the teacher submitted, general questions about the teacher's use of mathematical knowledge in their practice, and questions regarding how teachers believed their MKT developed. Most of the interview questions focused on the lesson plan, as it was expected that in-depth discussion of the lesson plans would reveal elements of MKT that teachers may not be able to answer in more general questions about their mathematical knowledge (Meade & McMeniman, 1992).

Analysis

All interviews were fully transcribed for analysis. A grounded theory approach to analysis was used in the style of Strauss and Corbin (1990).

Components of mathematical knowledge for teaching. After listening to the interviews and reading through the transcripts, initial codes were assigned to teachers' answers to two explicit questions about their MKT: "How did you apply your mathematics knowledge in this lesson?", and "Describe how your knowledge of mathematics has influenced your teaching." Next, full transcripts were coded for episodes that pointed to teachers' MKT, using the initial codes created from the responses to the two interview questions mentioned above as a non-exclusive framework. Like codes were organized to form categories using the constant-comparative method (Strauss & Corbin, 1990). Categories were refined and new codes and categories were formed when appropriate.

Next, lesson plans were revisited. Elements of the lesson plan which teachers discussed in their interviews were coded independently according to the categories developed from interview analysis. In most cases, the analysis of lesson plans supported the analysis of interview data. In cases where the lesson plans provided disconfirming evidence, codes and categories were revised to accommodate the data from the lesson plans or led to proposed explanations for why the disconfirming evidence existed (Creswell, 2007).

How knowledge developed. In a similar way to the coding of teachers' MKT, codes were assigned to passages in the transcripts that referred to the development of teachers' knowledge. All of these passages were explicit discussions of this development. As with the components of MKT, like codes were organized to form categories, and after revisiting transcripts, categories and codes were revised and refined as necessary. Lesson plans were not used as data for this part of the analysis.

Components of Mathematical Knowledge for Teaching

Teachers' MKT was categorized into five components. The teachers in this study valued: (a) building mathematical ideas using prior mathematics, (b) a range of examples of a concept, (c) appropriate applications of concepts, (d) connections between mathematical ideas, and (e) approaches to problem solving. Each of these components is defined in this section, and supporting examples are provided. Table 1 summarizes the results for this section. Note that it is not reported when teachers discussed a procedural knowledge of mathematics. That is, it is assumed that all math teachers, particularly exemplary teachers, have knowledge which includes knowing correct formulas for mathematical concepts, making calculations accurately, and representing functions as graphs and equations. In this section, more substantial components of MKT are reported.

Table 1
Components of Mathematical Knowledge for Teaching

Component of MKT	**Teachers**
Building mathematical ideas using prior mathematics *e.g., Using "completing the square" to derive the quadratic formula*	**11**
Range of examples of concepts	**8**
• Including "all" cases *e.g., Continuous and non-continuous periodic functions*	2
• Emphasizing various numerical values *e.g., Non-integer solutions to problems involving length*	4
• Providing counterexamples *e.g., Find numbers for x and y such that xy = 6, but ln(6) ≠ ln(x) + ln(y)*	3
• Creating intriguing examples *e.g., The graph of f(x) = x + sin(x)*	3
Applications of concepts *e.g., Using logarithms for the Richter scale*	**11**
Connections between mathematical ideas	**8**
• Between ideas in high school curriculum *e.g., Connecting combinatorics and the binomial theorem*	6
• Between high school and undergraduate mathematics *e.g., Knowledge of hyperbolic geometry to frame Euclidean geometry*	3
Approaches to problem solving *e.g., Using Heron's formula for area of a triangle when appropriate*	**8**

Building Mathematical Ideas using Prior Mathematics

All 11 teachers indicated that they thought it was important to build mathematical ideas in the classroom by grounding them in mathematics that students already understood. For example, Teacher 3 and Teacher 11 both drew on properties of exponents that students had already established in order to build understanding of properties of logarithms. Teacher 11 explicitly asked her students to justify properties of logarithms, such as $\ln(xy) = \ln(x) + \ln(y)$, in terms of properties of exponents (in this case, $e^a e^b = e^{(a+b)}$). She described the importance of using students' prior mathematics to deepen understanding:

> [Students have] seen logs before, but ninety percent of them have no idea why they do it. They have no idea what it means. They have no idea where it comes from. It's just something

> where someone taught them a process, they do the process, and they move on. They don't ever think about why it makes sense. And I feel like if I can show them why it makes sense and show them the math behind it, like that's why I like math.

In a similar way, Teacher 8 described how she used students' knowledge of "completing the square" to solve a quadratic equation in order to derive the quadratic formula:

> I really think that the most important thing is in the explanation of why . . . just where it comes from. . . . [When we study] the quadratic formula, [the students ask], "Well who invented that?" [Like] it just came out of the sky and someone just put these letters down. . . Completing the square is one way to factor, and it's not really that popular to teach it so much anymore because you can always use the quadratic formula. . . . You take that *ax* squared plus *bx* plus *c* equals zero, and you use your algebra on it and it becomes the quadratic formula. I always show the kids. This just didn't magically appear one day under the square roots. It came from something very simple that you know.

When discussing this piece of their knowledge, teachers discussed the fact that it was important for students to make sense of and remember mathematical ideas. Teacher 9 elaborated, "Don't make it something totally foreign. Make it something connecting back to what [the students have] already done. Because the more you can do that, the stronger it makes it, easier it makes it, the more likely they'll remember it."

A Range of Examples of Concepts

Eight teachers indicated that they know a range of examples that exemplify the extent of a concept and as well as counterexamples that show the limits of a concept. Teachers discussed four aspects of these examples.

"All" cases. Two teachers described how they wanted to include "all" cases of a particular concept. For example, Teacher 8 explained how she wanted students to see "all the different possibilities" of parabolas, including those with two real roots, those with one real root, and those with no real roots. She also wanted students to see relationships between the real roots, the complex roots, and the discriminant for each of these cases. Teacher 4 mentioned "cases" that were less common in the high school curriculum. When teaching periodic functions, Teacher 4 made sure to include examples that were not continuous, such as piecewise periodic functions, in order to stretch students' ideas of periodic functions. She discussed this planning process during the interview: "When I created the graphs, I wanted to make sure that I had representatives of all of the different cases that were going to come up."

Various numerical values. Four teachers used examples to emphasize the numerical values that were appropriate for a particular concept. One way that teachers did so was by trying to consistently include a range of real numbers in their examples (i.e., integers, rational numbers, and irrational numbers) as they were appropriate for the concept. Teacher 1 explained how, without encouraging students to consider several numerical values, they tend to focus on integers: "If I don't specify integers or real numbers, the students tend to focus on integers for some reason, and they don't think 'Oh yeah, a length can be 2.4.'" Similarly, Teacher 2 described how students often consider only integers when using non-calculus based methods to find the maximum value of a function. As a result, he often includes examples where the maximum value is not an integer in order to expand students' thinking.

Counterexamples. In order to exemplify the limits of particular concepts, three teachers discussed the importance of including counterexamples. For instance, Teacher 11 challenged her students to find numbers for x and y such that $xy = 6$, but $\ln(6) \neq \ln(x) + \ln(y)$. When discussing the triangle inequality in class, Teacher 1 asked students whether a triangle could have side lengths of 2, 3, and 5, for example. She explained that she included this because, "The [example] that students have the most trouble with is the one where the two shorter sides sum to exactly equal to the third side." She also asked students to find the possible values of x for a triangle whose side lengths are x, $2x + 1$, and $x - 1$. In this case, there are no such possible values for x.

Creating intriguing examples. Three teachers discussed how they created examples that that conveyed a particular idea by modifying other examples with which they were familiar. Teacher 6 described how she used parametric equations to create a graph that looked like a flower. Similarly, Teacher 4 discussed how she took the sum of the functions of $f(x) = x$ and $g(x) = \sin(x)$ in order to create a graph that students may not have seen before. Teacher 2 recognized that he could use a cubic function in a maximization application problem, as long as the application restricted the domain so that a maximum existed.

Applications of Concepts

All 11 teachers indicated that they know a range of appropriate applications for the mathematics they teach, and they found these applications useful for teaching. For example, Teacher 3 described the meaning of the Richter scale for measuring earthquakes and how the Richter scale relates to logarithms. Teacher 9 described how graphs with horizontal asymptotes can model the temperature of liquids cooling to room temperature, and Teacher 8 described how complex numbers are used to represent circuit elements in calculating net impedances. Teacher 11 also described how knowledge of applications of concepts was an important part of her MKT: "Because I have such a strong background in math and physics, I understand how the math is going to be used, and I can bring that into the classroom and help the kids see that as often as possible."

Teachers also explicitly discussed the importance of using applications with students. Teacher 10 described this idea:

> I try to find something that's relevant to what we're learning and try to put a realistic spin on it to just give [the students] understanding that it is used in the real world, and there is a reason that we're learning it.

It should be noted that, although teachers felt that knowing applications was an important part of their MKT, two teachers discussed how they did not believe that it was needed in every case. Teacher 5 discussed both the value of mathematical applications, as well as the fact that applications are not always appropriate. In fact, she cited a "negative impact of her knowledge" as knowing that students do not always apply the math they are learning; they just "use it to learn more math". Teacher 11 had a very similar sentiment:

> That's one of the hard things about teaching precalc. I feel like there's not a whole lot of quality applications that I can show the kids because the applications are either way too simple or they're not going to be able to do them until they're in calculus. And so, I try and explain that to them, and then eventually I say, "You just do it because you like math."

Connections between Mathematical Ideas

Eight teachers indicated they were able to see connections between different concepts and bring these connections into their teaching. This knowledge manifested itself in two ways: teachers made connections across the high school curriculum, or teachers made connections to advanced mathematics. It should be noted that this component of MKT is viewed as distinct from that of building mathematical ideas using prior mathematics. Teachers who demonstrated knowledge of the latter recognized a natural progression of mathematical ideas that helped students to build new ideas from previous ones. Teachers who demonstrated knowledge of connections between mathematical ideas were able to see relationships between mathematical concepts from different domains of mathematics, and they claimed to “bring in” connections when opportunities arose with the intention of broadening students’ knowledge rather than deepening it.

Connections across high school. Six teachers discussed the importance of making connections across the high school curriculum. Teacher 2 described how he connected the binomial theorem to combinatorics in one of his lessons:

> Where you're expanding $(x + y)$ to the fifth. And you go through all the theory, . . . and then you can say, “Alright, well let me just show you a little separate problem. If you have five books and you take three at a time, . . . well that's the same kind of problem as getting the coefficients for the binomial expansion.” . . . The more knowledge the math teacher has, the more that they can make connections, they can bring several different ideas in.

Teacher 5 was teaching the definition of a function in Algebra I and discussing the “vertical line test”. When a student asked if there was a horizontal line test, she was able to direct the lesson towards a discussion of one-to-one functions because of her knowledge of Algebra II. Teacher 5 called this a “serendipitous moment” in the classroom. In the following passage, she described drawing on her content knowledge in order to make this extension:

> Because I've got a pretty broad knowledge of mathematics from having taught it at so many different levels, I'm able to pull things out of the air, like pulling in the horizontal line test and one-to-one functions in this last lesson, without thinking too hard about it.

Teacher 1 connected geometry and probability by giving students the following task: “Randomly break a stick into three pieces. What is the probability that the three pieces form a triangle?” Teacher 10 connected regression and integration by asking students to plot data points to represent the perimeter of a two dimensional figure, find regression equations for this set of data points, and then integrate to find the area of the figure.

Connections to advanced mathematics. Three teachers discussed connecting high school concepts to undergraduate or graduate-level mathematics in their teaching. Teacher 8 described how she applied her graduate work in mathematics to high school teaching:

> I think that the more math knowledge you have, the more connections you see. And I found that even getting my Master's in math, the courses were ridiculous. . . . But then some weird thing would come up in class where he would explain it a certain way, and I'd be like, “How did I never think of it that way before?” And then I was able to take whatever that was and apply it to what I was teaching, though obviously it wasn't the same.

Teacher 5 claimed that she drew on her knowledge of college statistics when teaching high school statistics in order to "bridge some of the gaps between my students' knowledge and what they needed to know and further study." In a similar way, Teacher 9 claimed that his knowledge of hyperbolic geometry helped him to frame high school geometry and leave students open to the possibility of looking at geometry from a non-Euclidean perspective.

Approaches to Problem Solving

Eight teachers indicated that they are aware of several approaches to problem solving, and they draw on particular approaches that are appropriate for the context or situation. For example, Teacher 1 gave students two sides of a triangle and asked students to find the length of the third side that would yield the maximum area. For this problem, the teacher recognized that the use of Heron's formula[5] for the area of a triangle was appropriate because students were working with side lengths of a triangle. When Teacher 5 was discussing the restriction of $f(x) = x^2$ that was invertible and the domain of the inverse of this restricted function, she use lists of points on the graph generated by a graphing calculator to help students quickly discuss the domain. When using optimization problems in their (non-calculus based) lessons, Teacher 2 and Teacher 10 used technology to help students determine maximum and minimum values. Teacher 2 described how he felt this was an essential part of his MKT: "Getting the exact decimal, and that's where the technology comes in. So, knowing how to use the graphing calculator is really where the knowledge comes in."

Teachers also discussed the need to develop several approaches to problem solving in order to be responsive to students in the classroom. Teacher 10 described this need:

> It's now coming up with a second approach to get this formula or third approach. . . . I guess that's where our expertise comes in. I've had kids come up with methods that just don't seem kosher at first. And then you just sit there and you think about it, and sometimes you really need to reach, but they've come up with something else that you just haven't seen before. And to be able to give credibility to what they're doing, . . . Where it may not be our comfort zone. We need to be able to reach outside the box ourselves and help them keep going in the direction that they feel comfortable going.

How Knowledge Developed

The ways in which teachers discussed the development of their MKT were organized into three broad categories. Teachers discussed the influence of (a) math and math education courses, (b) professional experiences, and (c) personal experiences. Teachers were specifically prompted during interviews to discuss their undergraduate math courses but were free to talk about any other perceived influences. Each of these is described below. Table 2 summarizes the results for this section.

Table 2
The Development of Teachers' Mathematical Knowledge for Teaching

Positive Influence for Development of MKT	Teachers
Math and Math Education Courses	11

[5] Heron's formula gives the area of a triangle, A, with side lengths a, b, and c in terms of its semi-perimeter, s: $A = (s\,(s-a)(s-b)(s-c))^{1/2}$

• Content in advanced mathematics	6
• Math/ math education teacher or professor	6
• Math education courses	4
Professional Experiences	**9**
• Teaching a variety of courses	6
• Math-specific professional development while teaching	3
• Working with students and learning how students think	2
• Discussions with and ideas from colleagues	4
• Taking a leadership role in their schools	2
• Applying for National Board Certification	4
Personal Experiences	**9**
• Research and self teaching	5
• Personal/ family interest in mathematics	4
• Struggle with mathematics	5

Math and Math Education Courses

There were three elements of math and math education courses that teachers discussed when they described how they developed MKT: (a) the content of advanced mathematics, (b) a particular math teacher/ professor or math education professor, and (c) the content of math education courses. It is important to note that teachers did not discuss all of these elements in positive ways, as described below.

Content in advanced mathematics. Five teachers specifically indicated that they did *not* believe their MKT came from advanced math content. Teacher 4 described how the math that she learned as an undergraduate did not help her to teach high school:

> [In college], you start with calculus, and you go beyond, and then you become a teacher. And I didn't teach calculus until my tenth year of teaching. I'm teaching and nobody taught me how to teach geometry. I went to non-Euclidean geometry, but I'm teaching Euclidean geometry. I did four years of math in college that I'm not even touching. Ok, is it good to have that and know what that's about? Great. But, man, I could have really used some in-depth study of what I'm teaching here.

Similarly, when I asked Teacher 7 to discuss ideas from her undergraduate math courses that helped her to teach, she could not name anything:

> Interviewer: “What specific ideas from college math have helped you to teach high school?”
>
> Teacher 7: “Nothing.”
>
> Interviewer: “Really?”
>
> Teacher 7: “Um, college math. Oh my God.”

Teacher 3 described the influence of her undergraduate math courses, saying, “Honestly, I would love to say my college work [in mathematics] prepared me for [teaching], but I really don't think that it did.”

Six teachers did believe that the content of advanced mathematics helped them to develop MKT. Teacher 9 discussed how his undergraduate degree in math helped him understand high school math more deeply:

> It's kind of like making pizza. You have to stretch it out and when you let go it shrinks back. I think needed to see where things are going, even though I've come back and I'm doing algebra, but now I understand the algebra more. . . . I definitely think having the mathematics degree helps with those math skills.

Three teachers discussed how undergraduate mathematics helped them to understand what mathematics their students would study after high school so that they could frame their lessons appropriately. (See quotes from Teachers 5, 9, and 8 in Connections between Mathematical Ideas section.) Teacher 5 also valued the fact that she could answer students' questions about the types of courses they would take after calculus.

Other teachers mentioned a single, specific undergraduate course or idea that was helpful. For example, Teacher 1 mentioned the History of Math as the one valuable math course she took as an undergraduate. Teacher 11 said that studying vectors as an undergraduate helped her to understand how to teach matrices and vectors in the IB math course she was teaching.

Math/ math education teacher or professor. Six teachers described a math or math education teacher or professor that helped them develop MKT. These influential teachers ranged from middle school teachers to graduate-level professors. For example, Teacher 10 described one of his undergraduate instructors:

> I had a professor in college that really challenged me. … He was the hardest professor I've ever had. … I knew I was going to learn something in there, and he was going to challenge me. He developed me mathematically. … I think that has really helped me to be able to think about the methods that these kids are using, and kind of process what they're doing very quickly, and to say, 'Yes, you can do this,' or 'No, you can't do it.'

Interestingly, three teachers were graduates of the same Master's program and described the same math education professor as helping them to develop MKT. Teacher 2 described how this professor had elegant ways of motivating math concepts so they made sense. Teacher 2 also claimed that this professor influenced his belief that "The math teacher must know their stuff, they must know who they're stuffing, and they must know how to stuff them elegantly."

Math education courses. Five teachers felt that undergraduate math education courses were not helpful for their development of MKT. Three teachers believed that they needed teaching experience in order to benefit from methods courses. Teacher 9 described this idea:

> You're better off teaching basics and doing the pedagogy after you taught a year. … [Math education courses are] trying to teach refining skills, which you have no skills. So how do you refine it? … I think the education courses are wasted on the [young]. … Until you've done any of it, how do you keep refining when you're doing those classes?

Three teachers felt that the undergraduate math methods courses were not helpful to the development of their MKT in its current state because the methods learned in those courses were outdated.

Four teachers did feel that math education courses or math courses specific to teachers helped them develop MKT, and these were courses taken at the graduate level, after teachers had experience in the classroom. For example, Teacher 7 described the nature of the course coupled with her teaching experience that helped her to develop MKT: “In all my classes, which were all math classes, they're like, ‘The teachers in here, can you take this back to the classroom? Can you work with this?’” Similarly, Teacher 3 described how her graduate courses were helpful to her teaching:

> I do feel like for the most part, that was related more to the classroom, as far as the math involved in my Master's degree. Because I took problem solving in mathematics, which was great. . . . I took teaching geometry and teaching algebra in the middle school, so it was things that I could use.

Teacher 10 described how, in a graduate math course for teachers, he noticed ways in which the professor was explaining new ideas that he thought he could use in the high school classroom.

Professional Experiences

Nine teachers believed that their MKT developed through teaching mathematics as well as other professional experiences and opportunities, including (a) teaching a variety of courses (six teachers), (b) math-specific professional development while teaching (three teachers), (c) working with students and learning how students think (two teachers), (d) discussions with and ideas from colleagues (four teachers), (e) taking a leadership role in their schools (two teachers), and (f) applying for National Board Certification (four teachers).

Teaching a variety of courses. Six teachers specifically discussed how teaching a variety of courses helped them to develop MKT. Teacher 1 described this learning:

> One of the things that I think has been really helpful for me as a math teacher is having taught a variety of courses. I understand the connections between, the Geometry and the Algebra II, and where is this leading into Precalculus. I don't think I would have necessarily gotten that from my math degree in and of itself. I think the fact that I had taught a variety of courses allows me to see.

Teacher 8 also described how her knowledge developed from teaching a variety of courses:

> This was an Algebra II class. I found that teaching Algebra I and then teaching Precalculus really gave insight when you see what comes before, what comes after, what's the really important part. . . . The fact of always learning yourself and seeing. Again, I really think the more broad type classes that you teach, the better teacher it makes you.

Math-specific professional development. Three teachers believed that their MKT developed through math-specific professional development programs. For example, when asked where she gained the mathematical knowledge needed to teach, Teacher 1 replied:

> [I] go to a lot of workshops and things like that, professional development. Not so much courses that are designed for education in general, but specifically for math teachers. That has been very helpful. I definitely have learned things there that I would not have thought about otherwise.

Similarly, when asked how her knowledge of mathematics influenced her teaching, Teacher 3 explained how the math-specific professional development that she received made an impact:

> I really got some really great training. . . . I felt like at those workshops that I went to, . . . they were able to focus in on the different content areas. And I think getting some math-specific help for me—I mean at that point, I had been teaching five or six years, so I had my feet on the ground. I knew what I was doing, but it was the time for me to really kind of up my ballgame, I guess. . . . It was probably where I was in my teaching career, combined with everything else; it was just what I needed to kind of help me.

Colleagues. Four teachers discussed how they learned MKT from their colleagues. Teacher 2 described how he learned high school mathematics in a deep way from visiting the classrooms of other teachers: "We've had some great teachers, master teachers, here. You just say, 'Look, I'm free period 3, you're teaching calculus period 3, I'm there. Every day.' … I've done that more than anybody here." Teacher 4 felt like her school was particularly collegial so that teachers often discussed the mathematics they were teaching in depth.

Personal Experiences

Nine teachers discussed personal experiences as being influential to their development of MKT. Teachers described three general ways in which this occurred: through self-directed learning and research, through personality characteristics and personal background factors, and through a personal struggle with learning mathematics.

Self-directed research and learning. Five teachers sought out resources and researched new ways to think about math, or taught themselves mathematics. For example, Teacher 3 described, "I do a lot of research as far as how I'm going to teach a lesson, looking at different examples and different ways of how information is presented. So I do try to find some interesting ways or what worked." Teacher 5 described how she developed MKT by teaching herself mathematics:

> I actually got a book at the public library and taught myself college algebra and trig, and CLEPed[6] out, and did the same thing with Calc I and Calc II. So some of the math and the approaches to math that I use with my students comes from having to teach it to myself and do that independent sense making. Some of the approaches I take are kind of non-traditional, because sitting at my dining room table, I had to make sense of the formula, or the limit definition of the derivative. So I kind of had to step through it myself, and that's influenced a lot of my teaching.

Personal interest and background. Four teachers mentioned personal interests and background as helping them to develop MKT. For example, Teacher 9 described that his family encouraged mathematical and logical thinking when he was growing up, so he became naturally interested in math. Teacher 10 described that he had a passion for mathematics so that he in interesting in learning it more deeply.

[6] CLEP (College Level Examination Program) offers assessments that measure basic college-level knowledge. Many U.S. tertiary institutions offer course credit for passing these exams.

Struggle with mathematics. Personal experiences that led to development of MKT were not always positive ones for teachers. Five teachers believed that their own struggles with mathematics helped them to develop MKT. Some teachers struggled with mathematics as early as middle school, while others did not struggle until taking graduate math courses. Teacher 7 described how her struggle with undergraduate mathematics developed her MKT:

> When I was first studying mathematics, . . . I didn't get every single proof. I'd chew on it forever and I always didn't get it. . . . So, I guess as I studied mathematics, and I knew where the pitfalls were and how I overcame them, I passed that along to my students.

Discussion

The exemplary teachers in this study identified five components of mathematical knowledge that they believed were necessary for effective teaching: (a) building mathematical ideas using prior mathematics, (b) a range of examples of concepts, (c) connections between mathematical ideas, (d) applications, and (e) approaches to problem solving. Although all of these components are ones that many mathematics teacher education programs strive to develop in prospective teachers, the exemplary teachers in this study cited university coursework as only one source of their mathematical knowledge. Teachers in this study attributed much of their development of MKT to professional experiences as well as personal experiences. These results are similar to those of Ma (1999). Some researchers have argued that extensive undergraduate coursework in pure mathematics may not be the best way to prepare secondary math teachers, but these arguments are often accompanied by claims that preservice teachers could benefit instead from studying mathematics from a teacher's perspective (e.g., Even, 1990; Moreira & David, 2008; Potari, et al., 2007). However, 5 of the 11 teachers in this study did not believe that math education courses taken at the undergraduate level were helpful to their development on MKT. Teachers felt that these courses quickly became outdated or that they needed classroom experience in order to benefit from mathematics education courses. In his chapter of the *Second International Handbook of Mathematics Education* (2003), Stephens echoes this sentiment:

> It is unrealistic to expect that "deep understanding" or "pedagogical content knowledge" can be achieved by the conclusion of initial training. That kind of knowledge is sometimes referred to as case knowledge or situated knowledge. It is the sort of knowledge that cannot be learned from textbooks or at best is poorly learned from textbooks. It is a knowledge that grows out of reflective practice by a professional over a number of years. (p. 773)

Like Stephens, teachers believed that their MKT developed throughout their professional career and as a result of teaching itself as well as their personal motivation to learn more about the mathematics they were teaching.

Limitations

Although this study offers intriguing insight into the MKT of exemplary high school teachers, some limitations should be noted.

First, the participants for this study were chosen because they had received at least one of three major awards for their teaching. Mathematicians and mathematics educators are among the selection committee members for each of these three awards. As a result, the selection committees may be seeking teachers with components of MKT that the committee values, so the fact that the exemplary teachers in this study value many of the same components of MKT as mathematics educators should not be too surprising.

Second, this study seeks only the *perspectives* of exemplary teachers regarding their MKT. There is a possibility other subject-matter components of MKT are central to the practice of the teachers in this study, but they were not aware of them. Indeed, many researchers have argued that elements of MKT may be tacit, even for the most reflective teachers (e.g., Kennedy, 1998; Zazkis & Leikin, 2010). By the same token, there may be some components of MKT reported by teachers that are not actually useful for their practice.

Third, the components reported in this study are certainly not exhaustive. Teachers in this study cited other, more pedagogically-focused components of MKT, such as knowledge about mathematical topics that are difficult for students and methods that are engaging for students. These components were not reported here, since the aim of this study was to explore subject-matter components. Looking at the data from a pedagogical standpoint could yield different components that the ones developed here.

Fourth, although this study focused on MKT, teachers sometimes discussed general pedagogical knowledge that they found essential to their practice, including having strong relationships with parents or appropriate levels of discipline in the classroom. Hence, neither this study nor the teachers' interviews suggest that strong MKT is sufficient for successful classroom teaching. Indeed, exemplary teaching involves the intersection of many types of knowledge and expertise (Collinson, 1994; Potari, et al., 2007).

Significance and Future Directions

Research on subject-specific components of MKT has often taken a researcher's perspective and/ or been focused at the elementary level. Since the participants in this study are drawn from an exemplary pool of teachers, this study adds to the empirical research base on teachers' MKT and gives valuable insight into the components of MKT that are necessary for successful math teaching at the secondary level. Future research is needed to see if and how these components are actually carried out in exemplary teaching. This could be explored through observation and qualitative analysis of teachers' classroom practices. Indeed, Teacher 2 suggested that I observe his teaching in order to see his expertise at work.

This study also gives insight into how exemplary teachers believe their MKT was developed. Teachers felt that their MKT was largely developed through professional experiences. If these teachers' beliefs about this development are correct, then these results may imply that teachers need extensive opportunities for subject-specific professional development throughout their career. Preservice teacher education could help teachers to prepare for career-long learning by helping teachers to develop habits of reflection on the mathematics they are teaching. More research is needed to determine specific aspects of preservice education and undergraduate-level mathematics could be helpful for teachers.

Finally, in their interviews, teachers gave hints of which types of experiences may have helped to develop particular components of their MKT. For example, Teacher 1, Teacher 5, and Teacher 8 believed that teaching a variety of courses specifically helped them to make connections between different topics in high school mathematics. Teacher 5 believed that some of her "non-traditional approaches" came from teaching herself mathematics, and Teacher 11 believed that her knowledge of applications for mathematical concepts came from her coursework as an undergraduate. Future research could investigate teachers' opinions as to which types of experiences led to the development of particular types of knowledge.

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