How Intuition and Language Use Relate to Students' Understanding of Span and Linear Independence

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This report describes a case study in an undergraduate elementary linear algebra class about the relationship between students' understanding of span and linear independence and their intuition and language use. The study participants were seven students with a range of understanding levels. Findings indicated that students were more likely to successfully solve problems and explain concepts involving span and linear independence if they had low levels of self-evident intuitions and were better able to communicate their thinking in writing. Examples of students' interfering intuitions regarding span and linear independence are described. The report also includes possible instructional implications.

Keywords: Intuition, Language use, Linear algebra, Linear independence, Span

In an essay about his experiences teaching linear algebra, David Carlson (1997) posed a question that has become emblematic of students' learning in linear algebra: Must the fog always roll in? This question, he writes,

> refers to something that seems to happen whenever I teach linear algebra. My students first learn how to solve systems of linear equations, and how to calculate products of matrices. These are easy for them. But when we get to subspaces, spanning, and linear independence, my students become confused and disoriented. It is as if a heavy fog rolled over them, and they cannot see where they are or where they are going. (p. 39)

Research into the teaching and learning of linear algebra has spanned several decades, but the issue of how to clear the fog for students is still outstanding. In this report, I describe a research study designed to contribute to the understanding of how students learn concepts in linear algebra.

In his epilogue of *Advanced Mathematical Thinking*, Tall (1991) noted that many of the book's contributors believed students' difficulties in learning advanced mathematics could be explained by the discrepancies between the way students viewed mathematics and classroom instruction, which is often based on the formal structure of mathematics. More recently, in their discussion of advanced mathematical thinking, Mamona-Downs and Downs (2002) suggested traditional teaching of mathematics does not "connect with the students' need to develop their own intuitions and ways of thinking" (p. 170). An impediment to developing instructional theory based on students' intuitions is an incomplete understanding of how people develop abstract mathematical knowledge. Pegg and Tall (2005) compared several theories of concept development and derived a fundamental cycle of concept construction underlying each of the theories. However, there is no consensus on the mechanism of how this concept development occurs.

Some evidence exists to suggest language may play a role in concept development (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Devlin, 2000). Pugalee (2007) contends "language and competence in mathematics are not separable" (p. 1). MacGregor and Price (1999) and Boero, Douek, and Ferrari (2002) believe that metalinguistic awareness is necessary for students to coordinate the various notation systems in mathematics. Yet, little research exists that explores the relationship between students' language abilities and mathematics learning (Barwell, 2005; Huang & Normandia, 2007; MacGregor & Price). Interestingly, though, just as mathematics education researchers have found a contrast between intuitive thinking and formal mathematics, language researchers have found this same contrast between everyday language use and the demands of formal school language (Schleppegrell, 2001, 2007). It is possible, then, that language plays an important role in how students move from intuitive, everyday thinking to understanding formal mathematical concepts and theory.

The purpose of this study was to address two outstanding issues in the learning of advanced mathematics. The first issue is a theoretical difference between the ways in which students learn "naturally" and the formal structure of mathematics, and how this difference may or may not influence students' mathematical understanding. The second issue is the relationship between students' language use and their mathematical understanding. In particular, I elected to study these issues in the context of linear algebra. My research question was:

> How do students' intuition and language use relate to the nature of their understanding of span and linear independence in an elementary linear algebra class?

Theoretical Perspective

The theoretical perspective of learning for this research was the *emergent perspective* described by Cobb and Yackel (Cobb, 1995; Cobb & Yackel, 1996; Yackel & Cobb, 1996). The emergent perspective is a type of social constructivism that coordinates the social and psychological (individual) views (Cobb & Yackel). The *interactionist* view of classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988) represents the social perspective, while a constructivist view of individuals' (both students and teacher) activity (von Glasersfeld, 1984, 1987) represents the psychological perspective. The study's methodology reflected this theoretical perspective in that the social environment of the classroom was used as a lens through which to analyze individual student's work.

I drew from the literature for theories about intuition and language use. Fischbein (1987), Torff and Sternberg (2001), and Wilson (2002) all provide definitions of intuition (or the "adaptive unconscious" in the case of Wilson's work) that characterize intuition as an unconscious process that influences thought and behavior. Wilson and Fischbein postulate slightly different reasons for the existence of intuition in people, but common to both their reasons is the idea that intuition helps make sense of and organize people's incoming perceptual information so they can operate meaningfully in the world. Wilson documents some general characteristics of intuitions. People tend to regard their intuitions as self-evident. This is related to the tendency of early experiences to dominate later experiences in the formation of intuitions. Once formed, intuitions tend to be durable and resistant to change.

A key role of intuition is implicit learning, which results in tacit knowledge (Frade, da Rocha, & Falcão, 2008; Torff & Sternberg; Wilson). Ernest (1998a, b), in his model of mathematical knowledge, suggests tacit mathematical competences include knowledge-use of mathematical language, procedures, and strategies, meta-mathematical views, and personal beliefs about mathematics. There is little doubt intuition plays a role in students' classroom learning – both positively and negatively – although the exact nature of this role is still unknown (Torff & Sternberg, 2001).

As with intuition theory, the theoretical foundations explaining the role of language in learning are not well established. However, existing theory does suggest linkages between language, learning, and intuition. At a minimum, language may facilitate learning, and it may be that some in some cases, as with advanced mathematics, it is necessary for learning (Ferrari, 2004).

The role of language in learning plausibly fits within the emergent perspective. In the psychological perspective, language may mediate students' explicit and tacit knowledge construction (Leron & Hazzan, 2006) and in the social perspective, language is an important tool for developing mathematical meaning (Sfard, 2001).

Setting, Participants, and Data Collection

This research was a case study with cases drawn from one elementary linear algebra class with 28 students. Broadly, the unit of analysis for this study was individual students. However, in alignment with my research question, I focused my analysis on students' understanding of span and linear independence and on their intuition and language use related to these understandings. Using homework sets and the first exam, I evaluated the overall level of the 22 students who volunteered to participate in the study. Then, from the pool of students who consistently attended class and turned in their homework, I selected seven students to represent individual cases. The seven case students represented maximal variation in understanding levels among the students in the class.

I collected two types of data: instruction environment data and individual student data. The instruction environment data provided the contextual information. These data consisted of classroom field observations, classroom video tapes, instructional artifacts, and the course textbook, which was Lay's (2003) *Linear Algebra and Its Applications*. I attended each class in order to observe and record the classroom interactions. This provided me with an understanding of the instruction context of students' learning. From these data I determined instruction students received about specific problems as well as the classroom norms.

The individual student data provided information for analyzing the case students' understanding, language use, and intuition. The three types of individual student work were: written work consisting of homework and exams, journals, and interviews. There were 10 homework assignments, three mid-term exams, and the final exam. The homework problems were primarily from the course textbook, but occasionally the instructor assigned supplemental problems. Over the semester, the instructor assigned six journals in which students were asked to provide an overview of the course content since the previous journal and explain what difficulties, if any, they were having.

I conducted two interviews with each of the case students. The first interview was around the fifth week of the semester, which was a week after the first exam, and the second interview was one to two weeks prior to the final exam. The interviews were semi-structured (Merriam, 1998) as some interview questions were prepared in advance, but others were based on students' responses. I focused the first interview on students' initial understanding of span and linear independence. For the second interview, I focused on students' understanding of span and linear independence in the context of vector spaces and bases. Each interview lasted about an hour.

Data Analysis

In order to analyze the data with respect to the research question, I needed to operationalize the theoretical constructs of understanding, language use, and intuition. There were no generally accepted data analysis methods for these constructs, so I developed methods based on my literature review and the nature of my data. Following is a discussion of the methods I used to analyze and measure students' understanding, intuition, and language use.

Understanding Data Analysis

My goal in analyzing students' understanding was to be able to compare the nature and quality of understanding between students. I concluded there were two essential elements of understanding: definitional understanding and problem solving ability. I drew from Zandieh's (2000) understanding framework to develop definitional concepts. These are concepts embedded in the definitions of span and linear independence, so are necessary to understand these definitions. For instance, linear combination is a concept embedded in both definitions. One cannot conceptually understand the span of a set of vectors without knowing the role of linear combinations in constructing a span. Similarly, one cannot conceptually understand linear independence unless he or she understands the role of linear combinations of vectors is establishing linear dependence. Solution is another concept embedded in the definitions of span and linear independence. Determining linear independence is based on the solution of the homogeneous equation. For span, the existence of a solution to a linear system determines if a vector is in the span of a given set of vectors.

I expanded the notion of definitional concepts beyond concepts embedded in the definitions. I also included how well students understood the nature of the objects associated with the definitions. For example, students were not always clear what objects were linearly independent, sometimes referring to a matrix as being linearly independent, rather than the columns vectors of the matrix. With span, students sometimes mischaracterized the objects by specifying the vector in the span of a given set of vectors as the object creating the span. The other notion I included in definitional concepts was being able to understand how matrix operations related to the definitions. This included whether students associated pivot rows with the span of a set of vectors and pivot columns with the linear independence of a set of vectors.

I decomposed the second component of understanding, problem solving, into several dimensions that I derived from existing literature (Kannemeyer, 2005; National Resource Council, 2001) and the data. These were strategic competence, procedural fluency, work interpretation, and conceptual understanding. Strategic competence indicates the students' ability to select an appropriate strategy or heuristic to solve the problem. Procedural fluency indicates how successfully a student completes the procedures he or she selected to solve the problem. Work interpretation indicates how well the student interpreted the results of their computations with respect to what the problem is asking. Conceptual understanding indicates the degree to which the student appeared to understand the mathematical conceptions embedded in the problem or question. See Table 1 for a summary of the dimensions of understanding.

The data analysis for understanding consisted of coding the problems involving span or linear independence on the students' homework and exams. I used a four-level rubric to represent each dimension of definitional understanding and problem solving. A level 4 meant the data contained evidence that the student had competent, error-free knowledge with respect to a dimension. A level 3 meant the student work contained no evidence of problematic understanding or knowledge, but neither was there sufficient evidence to conclude competent, error-free knowledge. A level 2 meant there was evidence of both correct and incorrect understandings, and a level 1 meant there were significant problems with respect to the dimension.

I coded each student's work based on how he or she solved the problem therefore for a given problem I did not always code the same dimensions of understanding for all students. Table 2 contains the coding for a problem in which students were given two vectors from R^3 and asked to determine by inspection if they were linearly independent. (The vectors were multiples of each other, so were not linearly independent.) The dimensions that were relevant to the students' work were coded with one of the four rubric levels. The other dimensions were coded with an 'x'. For most procedural problems, such as the one in Table 2, conceptual understanding (CU) was not coded. In contrast, true/false problems that required an explanation typically were coded only for conceptual understanding and not coded for strategic competence (SC), procedural fluency (PF), or work interpretation (WI).

Table 1. Understanding Dimensions for Coding

Problem Solving Dimensions	Definitional Understanding Dimensions
SC – Strategic competence	M – Matrix operations
PF – Procedural fluency	S – Solution
WI – Work Interpretation	L – Linear combination
CU – Conceptual Understanding	O – Object reference

Table 2. Sample Coding for Understanding

Problem: Given 2 vectors from R^3, determine by inspection if they are linearly independent.									
Student	Student Response	Problem Solving				Definitional Understanding			
		SC	PF	WI	CU	M	S	L	O
1	The vectors are dependent because the[y] are multiples of each other.	4	4	4	x	x	x	4	4
2	*Does row reduction with augmented matrix, but interprets the augmented column as x_3.* No, by definition the LD, with x_2 being a free variable there are more than one possible solutions. When referring to LI there is only one unique solution.	1	4	2	x	4	3	x	x

Language Use Data Analysis

I found only one commonly used tool for analyzing the quality of students' writing: the 6-trait assessment rubric (Spandel, 2009). The purpose of the 6-trait rubric is to assess writing quality that consists of one or more paragraphs. The rubric contains six dimensions: ideas, organization, word choice, voice, sentence structure, and conventions. The student writing in my data typically consisted of at most a few sentences making it difficult to ascertain multiple quality dimensions in the writing. Therefore, I used a holistic score to measure the quality of students' written explanations. This score is a composition of three aspects of the 6-trait rubric: ideas, word choice, and sentence structure. I refer to this score as the *understandability* of the explanation. Note that this score was not related to the mathematical correctness of a students' explanation; it was related only to how well the student communicated their thinking.

To be consistent with the understanding rubrics, I used a four-level rubric to code the understandability of students' written explanations on homework and exams. The four levels were:

4 – The explanation uses math vocabulary appropriately, is clear and complete, and clearly relates to the problem.

3 – The explanation is not a 4, but the problems do not significantly interfere with the reader's ability to interpret the meaning of the explanation.

2 – The explanation has more significant problems in terms of understandability, completeness, and/or vocabulary use.

1 – The explanation is very difficult or impossible to understand.
If students wrote computational or other mathematical work, I interpreted the written explanations in light of this work.

Besides coding for understandability, I also recorded other characteristics of the students' writing based on systemic functional linguistic (SFL). The SFL literature contains descriptions of the lexical, grammatical, and content differences between informal and formal writing (Schleppegrell, 2001, 2007). Because the issues involved in helping students move from informal writing to formal writing seemed to mirror the issues involved in helping students build formal mathematical understanding from their intuitions, I decided to use characteristics of informal writing as indicators of the quality of students' language use. From the characteristics described in the literature, I selected those that commonly occurred in my data. These were mathematics vocabulary use, sentence grammar, and logical completeness. In addition, because lexical density (a measure of the frequency of content-specific vocabulary use) is often a differentiating feature between informal and formal writing (Schleppegrell), I decided to measure lexical density as well. I coded for the existence or non-existence of problems within each characteristic and use qualitative analysis to describe the nature of the problems. Mathematics vocabulary problems included referencing an incorrect object type for the specified process, using an incorrect form of the vocabulary word, and using the word in an incorrect context. Sentence grammar problems included incomplete sentences, run-on sentences, using pronouns with unclear referents, and atypical or awkward phrasing. Completeness problems included expressing ideas that were not clearly linked and not relating the explanation to the problem being solved.

Table 3 contains a sample of coding for language use. The 'Ex' column contains the code for understandability of the students' explanation based on the 4-level rubric. The 'V' (vocabulary use), 'S' (sentence grammar), and 'C' (completeness) columns contain check marks if there are problems with these characteristics in the students' writing.

Table 3. Sample Coding for Language Use

Problem: Given 2 vectors from R^3, determine by inspection if they are linearly independent.					
Student	Student Response	Ex	V	S	C
1	The vectors are dependent because the[y] are multiples of each other.	4			
2	No, by definition the LD, with x2 being a free variable there are more than one possible solutions. When referring to LI there is only one unique solution.	3		✓	
3	These vectors can be LD because there [?] 3/2*v1 is a weight of v2.	2	✓		✓

Understanding and Language Use Descriptive Coding

Once the data coding of the homework and exam problems was complete for the dimension of understanding and the understandability of language, I averaged the numerical assignments across problems for each student. Then I assigned descriptive labels to number ranges as shown in Table 4. For example, I assigned a 'Strong' label to averages that were 3.7 or higher.

Table 4. Category Labels for Definitional Concept Score Ranges

Category	Score Range
Strong	3.7 – 4.0
Good	3.3 – 3.6
Fair	2.7 – 3.2
Weak	2.3 – 2.6
Poor	< 2.3

Intuition Data Analysis

The students' homework and exam problems were the primary sources for the understanding and language use analysis while interviews served as the primary data for the intuition analysis. (The journals were used for both the understanding and intuition analysis.) For ascertaining intuition, I began by developing a list of characteristics of intuitive thinking from my literature review (see Table 5).

Table 5. Intuition Indicators

Intuition Indicators
1. The conception makes sense from an "everyday" perspective (Stavy & Tirosh, 2000).
2. The conception relates to initial learning experiences (Fischbein, 1987; Wilson, 2002).
3. The conception persists in the face of opportunities to change these conceptions (Fischbein; Vosnaidou, 2008; Wilson).
4. The conception builds upon on a prototypical example, rather than definitions (Sierpinska, 2000).
5. The conception is an overgeneralization of an example or concept (Ben-Zeev & Star, 2001; Fischbein).
6. The conception derives from a surface feature of a problem or representation, rather than the problem structure (Fischbein; Harel, 1999; Stavy & Tirsoh).

These characteristics were guideposts in my initial analysis, but I refined them based on my data. First, I elected to focus on intuitive thinking that appeared to interfere with students' developing mathematically correct understandings. Second, I classified students' intuition into two main categories, each of which had three sub-categories (see Table 6). These categories were self-evident indicators and surface indicators. The self-evident indicators align with the tendency of intuition to place more confidence in what "feels" right over pursuing more careful, rational analysis. The surface indicators align with the intuition's ability to quickly assess patterns and connections in the environment (Wilson, 2002). This ability, while important for being able to interpret one's environment in a timely manner, can also lead to eschewing the need to more purposefully engage in rational analysis.

Table 6. Intuition Indicators Used in Data Analysis

Self-evident Indicators	Surface Indicators
• Belief-intrusion	• Fuzzy-object
• Initial-learning	• Pseudo-definition
• Persistent-ideas	• Over-generalization

I found three types of self-evident indicators, which I labeled *belief-intrusion*, *initial-learning*, and *persistent-ideas*. Belief-intrusion occurred when the student held beliefs about the nature of mathematics that impaired their ability to comprehend mathematical concepts that felt contradictory to those beliefs. Initial-learning occurred when students' understanding appeared to be tied to initial learning experiences that subsequently interfered with correct conceptual development of ideas related to the initial learning. Persistent-ideas occurred when a students' incorrect understanding persisted even in the face of repeated exposures to contrary evidence.

I also found three types of surface indicators: *fuzzy-object*, *pseudo-definition*, and *over-generalization*. The fuzzy-object indicator represented when students did not understand or were not clear about what objects were related to definitions and procedures. This was characterized by students either incorrectly or inconsistently associating objects with definitions, such as thinking matrices were linearly independent rather than a set of vectors. The pseudo-definition indicator marked when students used procedures, abbreviated formal definitions, properties, or other ideas in lieu of formal definitions. The over-generalization indicator represented cases in which students' applied a concept or procedure in an inappropriate context. Pseudo-definitions could, in some cases, be considered a form of over-generalization. However, because of the importance of definitions in learning advanced mathematics, I distinguished between cases of over-generalization relating to definition and other cases of over-generalization. An example of a non-definition over-generalization was a student thinking that the variable m always represents the number of columns in a matrix even in the notation A_{nxm}.

This categorization of intuition was helpful in understanding the different ways in which intuitive thinking operated. However, not all patterns of thought were easily cast into a single category and sometimes indicators appeared to influence each other. Overall, though, this classification system was productive in providing a sense of how the students' intuitive thinking was similar and different.

I rated the level of each of these intuition indicators for each student as low, medium, or high based on how frequently the intuition indicators appeared in the students' data. A student with a low rating for an intuition indicator had no or few instances of that indicator interfering with the student developing mathematically correct understanding. In contrast, a student with a high rating had frequent instances of that indicator interfering with their developing mathematically correct understanding.

Findings

Table 7 contains a summary of the ratings of each student's understanding, language use, and intuition. The understanding and language use labels ranged from weak to strong. The intuition labels were low, medium, and high, with low indicating a low occurrence of intuition indicators in a student's thinking. The intuition labels consist of two parts, which reflect the two aspects of intuition: self-evident intuition and surface intuition. Based on this summary, the short answer to this study's research question is that there appeared to be an association between the quality of students' language use and the quality of their understanding. That is, students who were better able to effectively communicate their thinking in writing generally exhibited better understanding of span and linear independence. There was also an association between the degree to which a student's cognition had intuitive indicators and the quality of his/her understanding. The more a student's thinking had intuitive characteristics, the less likely he or she was to successfully solve problems and explain concepts involving span and linear independence.

Table 7. Summary of Student Understanding, Language Use, and Intuition Levels

Student	Understanding	Language Use	Intuition[1]
Anna	Strong	Strong	Low/Low
Brad	Good	Good	Low/Med-Low
Brian	Good	Good	Low/Med-Low
Carl	Good/Fair	Fair	Low/Med-High
Delia	Fair	Fair	Medium/High
Diane	Fair	Fair	Medium/Med-High
Elaine	Weak	Weak	High/High

[1]Self-evident intuitions rating / Surface intuitions rating

Understanding, Language Use, and Intuition Findings

Definitional understanding and problem solving had differing influences on the variations in the students' overall understanding ratings. Students with good or strong understanding had competent problem solving skills, while those with fair or weak understanding were less competent with respect to the work interpretation and conceptual understanding dimensions of problem solving. However, students' definitional understanding more closely aligned with their overall understanding. The students' definitional understanding of span and linear independence was not significantly different. Students' understanding of matrix operations and object reference tended to be stronger than their understanding of solution and linear combination, although this pattern was more evident for linear independence than for span.

Students' writing patterns were relatively consistent over the semester. The quality of explanations was slightly better for linear independence problems than for span problems, which might be a reflection of students' generally higher confidence in their linear independence understanding as compared to their understanding of span. The explanation characteristics (vocabulary use, sentence structure, and completeness) were elements of consideration in evaluating the quality (understandability) of an explanation, but they were not determining factors. That is, the presence or absence of these characteristics did not dictate the quality level. However, completeness appeared to be associated with the explanation quality. For the good and strong explanations, the percent of explanations tagged as incomplete was no more than about 30%. For lesser equality explanations (fair, weak), the percent of incomplete explanations was about 40% or more.

Vocabulary use was weakly associated with explanation quality as it differentiated the strongest and weakest explanations, but was more constant in the middle levels (good, fair). Sentence structure did not appear to be associated with explanation quality. This characteristic related to grammar, such as the presence of run-on sentences. However, such issues did not always significantly interfere with the understandability of the explanation. Lexical density was not associated with explanation quality. In the literature, formal school writing tends to have a higher lexical density than informal writing. In this study, lexical density might not have differentiated explanation quality because 1) the explanations were relatively short (typically fewer than 30 words) and 2) mathematical words in students' explanations contributed to the lexical density of the explanation regardless of whether the students used them correctly.

The two types of intuition indicators were differently associated with students' quality of understanding. Students with good or strong understanding had low levels of self-evident intuitions, while students with fair or weak understanding had medium or high levels of self-evident intuitions. In contrast, the surface intuitions tended to more directly vary with understanding level. While medium or high levels of self-evident intuitions were associated with medium or high levels of surface intuitions, the converse was not true. Thus, it is possible that self-evident intuition causes or influences surface intuition, but that surface intuition can develop without strong self-evident intuition.

Examples of belief-intrusions were a philosophy of mathematics that conflicted with the philosophy of mathematics underlying the course and a belief that mathematical definitions should align with everyday definitions of words. Several of the students exhibited an initial-learning indicator when their conception of span was dominated by the first span theorem in the textbook, even though this only represented one aspect of span. Elaine's learning was impaired by her attachment to learning she acquired prior to the class. Overall, though, students did not exhibit significant levels of initial-learning intuition. In examining the persistent-ideas, I believe these ideas are more likely to be associated with learning that occurred prior to the class. For example, notions about R^n and variables tended to be persistent and were certainly encountered prior to the linear algebra course.

Pseudo-definitions were the most common type of surface indicator. This might have been related to three factors. First, students avoided the textbook definitions because they felt the definitions were too difficult to interpret. In particular, they found the notation difficult to follow. Second, some students did not believe that knowing definitions was important to the class, so focused their learning on the properties and procedures associated with span and linear independence and used these in lieu of formal definitions. Third, students were able to use their pseudo-definitions successfully in their written explanations. For example, Delia commonly wrote "one unique solution" as a justification for vectors being linearly independent even though she had no real understanding of what that phrase meant with respect to the formal definition of linear independence. Another common example was students equating trivial solution with a unique solution. When the students said the homogeneous equation has a trivial solution, it is true that it is both trivial and unique, so their understanding was never found wanting.

The most common fuzzy-object thinking was that linear independence related to a matrix rather than a set of vectors. This most likely stemmed from students working primarily with matrix operations in determining linear independence rather than the formal definition of linear independence. Students tended to be less clear about the objects associated with span than they were about the objects associated with linear independence. This might be because it is necessary to specify two objects when discussing span (e.g., a set of vectors spans a vector space, a vector is in the span of a set of vectors) and the order of the objects is important.

The types of over-generalizations students made varied. However, a couple of students mentioned it was difficult for them to distinguish between the definitions of terms and/or what different problems were asking. This might be an indication that their tendency to over-generalize interfered with their being able to differentiate definitions and problems. The students' abilities to interpret mathematical language might also be a mediating factor. Mathematical notation was sometimes a cause of over-generalization as was the case when Carl thought that span was in the definition of linear independence because he saw a vector equation that triggered his thinking about linear combinations, which he associated with span.

Span and Linear Independence Findings

Students' quality of understanding of span was the same or slightly poorer than their understanding of linear independence. However, in the course, students had more practice on homework problems with linear independence than with span. Even with equal practice opportunities, the findings suggest span might be a more difficult concept to learn than linear independence. Span has a more complex object structure, involving a set of vectors and a vector space, while linear independence involves only a set of vectors. Since object referencing was difficult for some students, this may interfere with the learning of span. Also, span can be viewed in reference to spanning an entire space or a subspace, a difference some students never mastered. Linguistically, *span* can be used in various forms, such as *spanned by* and *spans.* Students with poorer language skills may struggle to interpret these variations correctly.

Although there were differences in how well individual students understood span as compared to linear independence, these differences were never large. That is, a student never had a strong understanding of linear independence and a weak understanding of span. In addition, there was no significant difference between the understandability of the students' language use involving span as compared to their language use involving linear independence. It is possible this is due to the intuitive habits students brought to the learning of both concepts. If these habits were productive, they supported the learning of both concepts, while, in contrast, if these habits were not productive, they inhibited the learning of both concepts. Another explanation is that span and linear independence both have similar definitional concepts (solution and linear combination). How well students understood these definitional concepts likely influenced the quality of understanding for span and linear independence in a similar way.

For both span and linear independence, students tended to rely on procedural learning rather than conceptual learning. This appeared to be a function of two factors. First, some students preferred to learn mathematics procedurally and even felt uncomfortable approaching these concepts conceptually. This stemmed from beliefs students held about the nature of mathematics. Second, based on homework and test questions, most students believed that procedural skills were mostly sufficient for them to be successful in the class. The textbook played a role in this perception. Many problems in the textbook (Lay, 2003) could be done without much conceptual understanding. Thus, students' interpretation of the instruction environment led them to downplay conceptual learning. Sierpinska, Nnadozie, and Okta (2002) found the linear algebra courses they studied emphasized procedures over concepts as well. In their research they concluded, "The theory remained the frolic of the university professor" (p. 164).

Students could rarely state a formal definition of span or linear independence and, when given the definition, sometimes struggled to interpret it correctly. As with their pursuit of procedural learning, students concluded they did not need to know definitions. Another reason students did not learn the definitions was they found them to be too difficult to understand, mostly because they were not able to interpret the notation. This finding mirrors other research about students' learning of definitions in advanced mathematics courses (Edwards & Ward, 2004;

Vinner, 1991).

In this study students were not always clear on what object was linearly independent, sometimes thinking it was a matrix instead of a set of vectors. In concept development theory, students need to reify their procedural understanding in order to develop structural understanding. In the case of linear independence, students need to move from knowing linear independence as a process involving matrix operations to knowing linear independence as a characteristic of a set of vectors. It may be important that students reify foundational concepts in order to learn new concepts. According to Semadeni (2008), reification can lead to "deep intuition," which can then facilitate deductive thinking.

It seems possible that not knowing the objects associated with processes could interfere with reification of the process and thus impede students' developing structural understanding. For example, students who routinely ignore or misunderstand the objects associated with processes, may have a weak sense of what structural understanding in mathematics is because they focus their energies on procedures and algorithms and do not seek broader generalities and connections. If linear independence is simply determining if there is a pivot in every column of a matrix and this understanding is functional for the student, then the student does not consider linear independence as a characteristic of a set of vectors and so never reifies the concept of linear independence. Conversely, it may be that helping students understand the objects associated with processes could facilitate reification of those processes.

Discussion

This study generated preliminary evidence to suggest that students whose thinking habits are characterized by self-evident intuition indicators as defined in this research are less likely to develop mathematically correct understandings in linear algebra. Surface intuitions in the absence of self-evident intuitions were less detrimental to students' success in learning, although higher levels of these intuitions still hindered students' conceptual development. A more robust theory about what these intuitions are, how they influence learning, and how they can be remediated might help lead to improvements in advanced mathematics instruction.

Another finding in this study is that students' ability to write understandable explanations of their thinking, regardless of the mathematical correctness of those explanations, was associated with the quality of students' understanding. This suggests a link between students' language use and their mathematical learning. An implication of this finding may be that students' mathematical writing has some validity in assessing students' mathematical understanding. In addition, it may be that instruction that supports students' effective language use in mathematical explanations may help support students' mathematical learning.

The study has several limitations. Because it was conducted in a single class, the findings may have limited transferability. Also, the nature of the data sources (student work and student interviews) may have limited the validity of the findings. Future research may refine or extend this study's findings in other linear algebra classes. It may also be fruitful to explore this research question in other advanced mathematics classes, such as abstract algebra and analysis.

The findings suggest possible classroom implications. While none of the instructional methods are new, this research may underscore their validity in supporting students' learning of mathematics by reducing the role of interfering intuitions. Instructional recommendations include helping students develop metacognitive awareness (Fischbein, 1987) and implementing compare and contrast activities (Marzano, Pickering, & Pollock, 2001). Several researchers have outlined more elaborate instructional tools. These include the instructional practices developed by researchers studying the role of beliefs in mathematics (Muis, 2004), conceptual change in science and mathematics (Vosniadou & Vamvakoussi, 2006), and in reducing misconceptions in mathematics (Stavy & Tirosh, 2000). In order to help students develop their language skills, which in turn may support their mathematical learning, it may be helpful to provide opportunities for students to engage oral and written language practice.

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