FROM INTUITION TO RIGOR: CALCULUS STUDENTS' REINVENTION OF THE DEFINITION OF SEQUENCE CONVERGENCE

Michael Oehrtman
University of Northern Colorado
michael.oehrtman@unco.edu

Craig Swinyard
University of Portland
swinyard@up.edu

Jason Martin
Arizona State University
jason.h.martin@asu.edu

Catherine Hart-Weber
Arizona State University
catherine.hart@asu.edu

Kyeong Hah Roh
Arizona State University
kyeonghah.roh@asu.edu

Little research exists on the ways in which students may develop an understanding of formal limit definitions. We conducted a study to i) generate insights into how students might leverage their intuitive understandings of sequence convergence to construct a formal definition and ii) assess the extent to which a previously established approximation scheme may support students in constructing their definition. Our research is rooted in the theory of Realistic Mathematics Education and employed the methodology of guided reinvention in a teaching experiment. In three 90-minute sessions, two students, neither of whom had previously seen a formal definition of sequence convergence, constructed a rigorous definition using formal mathematical notation and quantification equivalent to the conventional definition. The students' use of an approximation scheme and concrete examples were both central to their progress, and each portion of their definition emerged in response to overcoming specific cognitive challenges.

Keywords: Limits, Definition, Guided Reinvention, Approximation, Examples

Introduction and Research Questions

A robust understanding of formal limit definitions is foundational for undergraduate mathematics students proceeding to upper-division analysis-based courses. Definitions of limits often serve as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally-quantified mathematical statements, and transitioning to abstract thinking. The majority of the literature on students' understanding of limits (e.g., Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992; Williams, 1991) describes informal student reasoning about limits, with particular attention given to a myriad of student misconceptions. However, there is a paucity of research on student reasoning about formal definitions of limits. The general consensus among the few studies in this area seems clear – calculus students have great difficulty reasoning coherently about the formal definition (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). What is less clear, however, is *how students do come to understand the formal definition*. Indeed, this is an open question with few empirical insights from research to inform it (Cottrill et al., 1996; Roh, 2008; Swinyard, in press). Oehrtman (2008) proposed an approach to developing the concepts in calculus through a conceptually accessible framework for limits in terms of approximation and error analysis. The design criteria of this approach are to maximize coherence across topics and representations and to establish a potential foundation for abstraction and formalization.

Students were recruited to participate in our study from a course that employed Oehrtman's approach. This report addresses the following research questions:

1. What are the cognitive challenges that students encounter during a process of guided reinvention of the formal definition for sequence convergence?
2. How do students' resolve these cognitive challenges, and how does that process help advance their mathematical thinking about limits of sequences?
3. To what extent and in what ways do these students use aspects of approximation and error analysis in their reinvention of the formal definition?
4. What implications can be drawn about the nature of students' and researchers' co-productive activity in the process of a guided reinvention?

Theoretical Perspective

We adopted a *developmental research* design. The purpose of developmental research is "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (Gravemeijer, 1998, p. 279). Task design was supported by the *guided reinvention* heuristic, rooted in the theory of Realistic Mathematics Education (Freudenthal, 1973). Guided reinvention is described by Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000) as "a process by which students formalize their informal understandings and intuitions" (p. 237). Faced with the alternative of having the students reason about a formal definition provided by us (i.e, a conventional definition from a textbook), we felt that engaging the students in the articulation of a personal concept definition (Tall & Vinner, 1981) would better position us to identify *authentic* challenges students experience as they formalize their intuitive understandings of sequence convergence. Such an approach has provided rich data in other studies about students' understandings of formal definitions in advanced calculus (Swinyard, in press). The design of the teaching experiment activities described in the methods below was inspired by the *proofs and refutations* design heuristic adapted by Larsen and Zandieh (2007) based on Lakatos' (1976) framework for historical mathematical discovery.

We employed Oehrtman's (2008) approximation framework for instruction of calculus to provide the rich informal foundations on which such formalization might be based. Oehrtman (2009) framed the necessary systematization of students' spontaneous concepts about approximations in terms of the dialectic interplay between everyday and scientific concepts in Vygotsky's (1987) zone of proximal development:

> Having already traveled the long path of development from below to above, everyday concepts have blazed the trail for the continued downward growth of scientific concepts; they have created the structures required for the emergence of the lower or more elementary characteristics of the scientific concept. In the same way, having covered a certain portion of the path from above to below, scientific concepts have blazed the trail for the development of everyday concepts. They have prepared the structural formations necessary for the mastery of the higher characteristics of the everyday concept. (p. 219).

In particular, Oehrtman (2008) engages students in activities that repeatedly develop the concepts in calculus defined in terms of limits through the consistent structure represented by viewing the limit (L) as an unknown value to be approximated, the argument of the limit (say a_n in the case of sequences) as the approximations, the magnitude of difference between the two ($|a_n - L|$) as the error, ε as the error bound ($| a_n - L | < \varepsilon$), and the domain process as allowing one to control the degree of accuracy ($n \to \infty$). Our expectation was that the class activities in developing such everyday concepts about approximation to limit structures would provide students with a significant benefit for the formal definition.

Methods

The authors conducted a six-day teaching experiment with two students at a large, southwest, urban university. Five students volunteered for the study and two average to slightly above average performing students, Megan and Belinda (pseudonyms), were selected for the teaching experiment based on our estimation of their ability to effectively work as a pair. Evidence of this ability was based on instructor observations of their communication skills, engagement in class material, and their familiarity with one another from in-class group work. The full teaching experiment was comprised of six 90-120 minute sessions with a pair of students currently taking a second-semester calculus course whose topics included sequences, series, and Taylor series. The central objective of the teaching experiment was for the students to generate rigorous definitions of *sequence*, *series*, and *pointwise convergence*. Megan and Belinda had not received instruction on these formal limit definitions, as confirmed by their current and prior instructors. The research reported here focuses on the evolution of the two students' definition of sequence convergence over the course of the first three sessions of the teaching experiment.

Activities commenced with students generating prototypical examples of sequences that converge to 5 and sequences that do not converge to 5. The majority of each session then consisted of the students' iterative refinement of a definition, with the aim of precisely characterizing sequence convergence. The students were to evaluate their own progress by determining whether their definition included all of the examples of convergent sequences and excluded all of the non-examples. We then guided them in making explicit the nature of any conflicts, hypothesizing several ways the conflicts might be resolved, then employing a chosen approach. This established a cyclic process of definition refinement (Figure 1), that resulted in 26 distinct written definitions over the first three sessions of the teaching experiment.

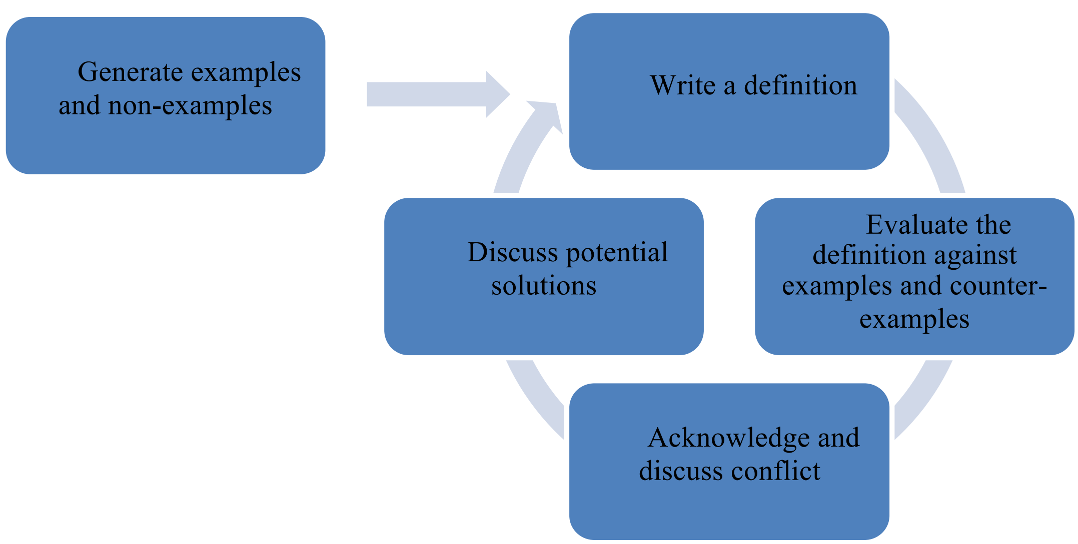

Figure 1. Iterative refinement in the process of guided reinvention of a formal definition.

Of the five-member research team, two researchers served as facilitators of the reinvention process, while the other three observed a live audio and video feed from a separate room. A live text chat between the two sets of researchers allowed the three observers to suggest to the two facilitators areas to probe the students or changes in direction for the protocols. The two researchers facilitating the guided reinvention always had discretion of where to focus the discussion. The entire research team debriefed immediately after each session, and then reviewed the video tapes to develop hypotheses about the progress of the guided reinvention, unforeseen challenges, and potential resolutions. The team then made adjustments to the protocols for the following day's session.

Starting the week after the teaching experiment sessions, the research team quickly transcribed all video tapes and created detailed content logs of each session. The logs consisted of time-stamped descriptions of the activity of the students and researchers with separate theoretical notes about how that activity was progressing toward the formal definition. We then coded the video timeline for what problem(s) the students were explicitly addressing. For each problem, we attempted to determine both the origin and the students' resolution if one occurred.

Results

We found it beneficial to distinguish between *problems*, which were those issues explicitly raised by the students most commonly as conflicts between their evoked concept image and their stated definition, and *problematic issues*, those that arose unbeknownst to the students as conflicts between the students' stated definition and the standard formal definition. Since problematic issues were not salient for the students, we often found it necessary for the facilitating researchers to prompt reflection on them. When successful, we note that the problematic issue became a problem for the students. In some cases this occurred without our intervention, as the students uncovered the problematic issues on their own. Whether prompted by the facilitating researchers or initiated by the students, we observed that resolution of problems always required explicit conversation focused intentionally and specifically on the problem. On the other hand, we also observed several instances in which problematic issues were resolved implicitly, in the background, while the focus of conversation was on resolving another problem (sometimes related while other times unrelated). In the following, we provide several examples of problematic issues, problems, and implicit and explicit resolutions while noting the interplay between the facilitating researchers and the students.

The examples and non-examples created during the preceding class session were prominently displayed on large sheets of paper on the wall near the table where Megan and Belinda were seated. These graphs were then constantly available for the facilitators to reference in questions and for the students to use in explanations and in testing their definition. The facilitators began by asking the students to work together to complete the statement "A sequence converges to 5 as $n\to\infty$ provided…" in a way that included all of the examples but excluded all of the non-examples. The first definition created by Megan and Belinda reflected a standard conception of always getting closer.

> Definition 1: A sequence converges to 5 as $n\to\infty$ provided n+1 is closer to 5 than n for any n value.

We note that their metonymic use of n+1 and n for a_{n+1} and a_n was clarified throughout their discussion and did not create any discernible confusion or miscommunication between them. They quickly reframed their definition in dynamic terms.

> Definition 2: A sequence converges to 5 as $n\to\infty$ provided that the number approaches or is 5 and no other number.

For both of these definitions, Megan and Belinda pointed out examples of sequences converging to 5 posted on the wall that were not monotonic early in the sequence and commented that their definitions did not apply to those. Their attention to this issue and expressed desire to change their definition to include sequences that might have "bad early behavior" constituted the first explicit problem to which the students attended. Six minutes into the session, Megan suggested, "What if we were to say after some point n it...We have random stuff going on [*pointing at all of the convergent graphs*], but at some point it does start doing the closer and closer and closer thing." Belinda immediately agreed and they incorporated the phrase "some point n" into their definition. Some version of this idea appeared in almost all subsequent definitions (exceptions appeared to be out of oversight and were usually changed to include the phrase). For example their third definition read,

Definition 3: As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5.

Although the students were happy with this resolution and it is reminiscent of the N in a standard ε-N definition, the researchers noted four problematic issues with their implementation of it. Specifically, i) their description of the "some point n" referred to where the sequence began exhibiting "convergent behavior" and as such ii) was a fixed location for a given sequence, iii) they used n to refer both to this point where the behavior changed and as the index on terms of the sequence, and iv) they did not capture precisely which values of n should have the property that a_{n+1} is closer to 5 than a_n. Schematically, this problem, solution, and emergence of new problematic issues is represented in Figure 2.

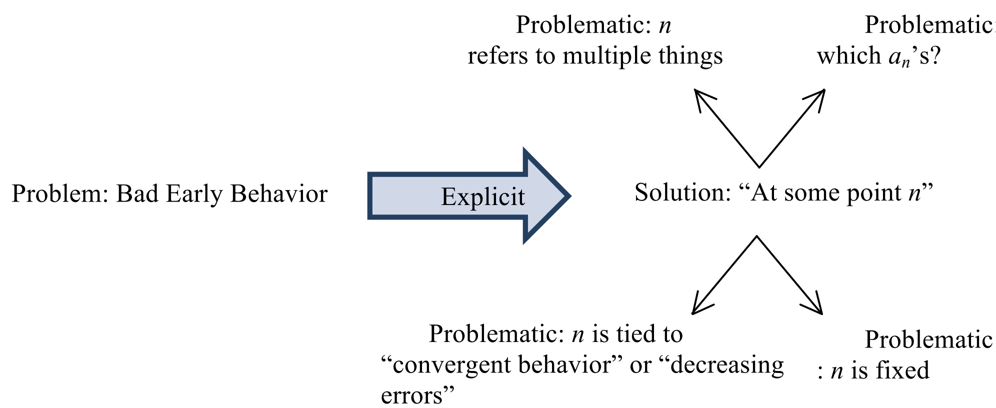

Figure 2. Megan and Belinda's first problem resolution created new problematic issues of which they remained unaware for a significant duration.

We now discuss how the first two of these problematic issues, shown in the lower region of Figure 2 were both resolved implicitly by Megan and Belinda. This means they focused on different explicit problems whose resolution also solved these problematic issues. In particular, when they started using language involving infinitesimals, Craig asked what they meant by "infinitely close."

Craig: Now both of you have mentioned this "infinitely close thing." Can you say a little bit more about what you mean by infinitely close? Or maybe is there a way that you could encompass that into your definition at all?
Megan: Negligibly small.
Belinda: Yeah, where an error exists, but it's…
Megan: Um… We were doing that stuff before where our approximation had an error. Remember when we had to find it within .0001, or whatever?

Belinda: Yeah.
Megan: I guess we can say "within a designated margin of error" or "designated error bound" because it would vary by what our error bound was for that particular problem or situation.

Thus, while trying to interpret the meaning of phrases like "infinitely close" and "negligibly small," they recalled ideas about bounding errors from their work in class. Later they were wrestling with the same issue and Megan explained,

> Well if an alternating [graph] never actually becomes 5 but we ***accept*** it as a convergent graph, um, that must mean at some point it gets within a range of 5 [*holds hands apart horizontally*] that we accept it as converging to 5 whereas say if we're looking at the sine graph [*moves hands farther apart*], it doesn't get within that, that distance. It stays out at –1 to 1…. The regular sine graph. It stays within –1 to 1. That distance is too great to call it convergent. But if [the alternating graph] never actually becomes 5 [*moves hands back together*], but it gets that small little range. There's gotta be a little acceptable range at some point that you say it's convergent [*holds two fingers close together horizontally*] even though it never actually becomes [*bounces fingers off to her right*].

At this point we see that Megan seamlessly transferred her use of "at some point" to refer to a location where the sequence enters an "acceptable range" of the limit, and the two produced Definition 12 shortly afterward.

Definition 12: A sequence converges to 5 as $n\to\infty$ i.f.f. $\exists\ n$ such that for all a_n $|5-a_n|\leq .01$

From this point on during the teaching experiment (with one brief exception) both students used the phrase "at some point" consistent with this interpretation. Consequently they dropped the previous usage tied to the beginning of "convergent behavior" or "decreasing errors," implicitly resolving that problematic issue. This is represented in the lower left of Figure 3.

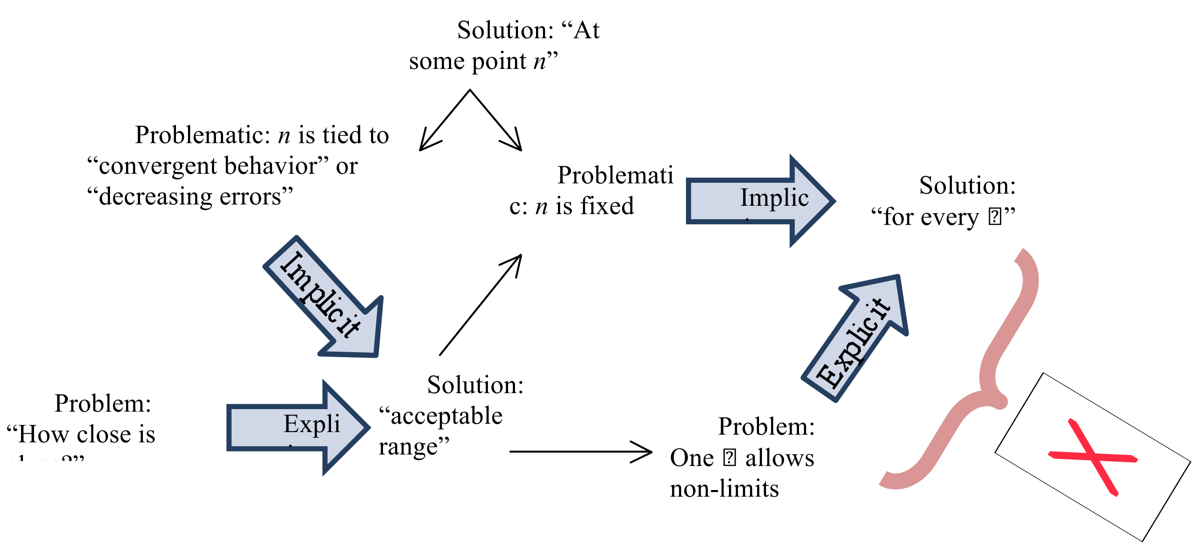

Figure 3. Implicit resolutions to two of Megan and Belinda's problematic issues.

We also note that the problematic issue of n being a fixed location remained in this new formulation. Specifically all four references to the "acceptable range" in Megan's quote above make clear that she viewed it as having a fixed size (not consistent with the universal quantification of ε in formal definition). Both Megan and Belinda spent the next hour of discussion clearly referring to the "acceptable range" in this way. In doing so they incorporated the idea of an "acceptable range" into their definition by either choosing some small number, as seen in Definition 12 or by replacing the number with an expression of an unknown quantity such as a question mark. The fixed nature of the "acceptable range" thus reinforced the problematic issue of "some point *n*" also being fixed. From this point on, some version of their "acceptable range" remained a part of their definition.

Megan and Belinda quickly realized that such a definition excluded some non-limits that were problems for their "approaching" formulation. Megan noted, "If we could say within this error bound of .0000ooo1 or something, you know, if it's within this range of this number [*holds hands horizontally close together*], then that way it won't be within that range of 6. It'll be within that range of 5." On the other hand, they also realized that no matter how small their "acceptable range" might be, it would still allow some non-examples to fit the definition, for example a sequence converging to 4.99999. For instance,

Megan: Instead of 01 we'd have like 001. Or like 10 zeros and a 1. Or something that would bring it closer and closer and closer to 5, but we're not talking about an infinite number of decimals possible…. So the only thing we'd have to figure out is the possibility that it would still be converging to some number within there.

Belinda: The "whatever range you want" part would exclude [4 as a limit] IF you made your range small enough.

Belinda's comment on the exclusion of a non-limit was emphatically conditioned on a sufficiently small choice of "your range." They argued that using specific numbers like .01 was "arbitrary" in the sense of not being intentionally or well-chosen. As Belinda expressed, "You could start changing the definition of convergence deciding on however, however close you want it to be at that day."

During this extended period of wrestling with the problem of a fixed "acceptable range," Megan and Belinda were also confused about the convergence of a damped oscillating sequence, as detailed by Hart-Weber, Oehrtman, Martin, Swinyard & Roh (2011). Specifically, despite introducing the idea of an "acceptable range," they also clung to their notion of requiring "decreasing errors." For example, their definition midway through the second session read,

Definition 14: A sequence converges to 5 as $n \to \infty$ only if there exists a point n after which the error does not exceed $|5-a_n| \leq .001$ and $|5-a_{n+1}| < |5-a_n|$.

Since, no matter how far out you go, the damped oscillation always has places violating the requirement of $|a_{n+1}-5| < |a_n-5|$, they felt it must diverge. While discussing this example, Megan described one property of the damped oscillation as, "it will eventually get it within whatever range we want to put it in though, whether it's point 25 zeros and a 1 or a thousand zeros and a 1, it will get within that range and it will stay there." Here Megan verbalized a nearly perfect informal statement of the limit definition including the universal quantification on ε that would resolve their problem of the "acceptable range" being fixed. Unfortunately she did so in the context of describing what she felt was a "divergent sequence," not in the context of addressing their problem of a fixed "acceptable range" allowing non-examples. Thus, while the students were at times explicitly focusing on the right problem and at other times expressing the needed solution, they did so in separate segments of conversation.

After wrestling with the problem of a fixed "acceptable range" for over an hour, the facilitating and observing researchers all agreed that the students had extensively explored the nature of the problem and had even expressed and understood an idea that would solve their problem. We felt it would be an appropriate time to intervene and suggest bringing their solution and potential problem into common relief. As Megan and Belinda were discussing how to resolve the issue that any acceptable range would necessarily allow extraneous limit candidates, Jason intervened with a suggestion.

Jason: What about "for ***every*** error?"
Belinda: So then that would work because ***every*** error, 'cause then it has to work within ***every*** error or it's not convergent. If it doesn't work within ***every*** error, then it's not convergent.
Megan: Yeah, that works.

At this point, Megan and Belinda immediately, although somewhat awkwardly, incorporated the idea into their definition.

Definition 21: A sequence converges to 5 as $n\to\infty$ only if there exists a value N after which $|5 - a_n|$ does not exceed every error after N.

From this point on, Megan and Belinda persisted in using a universal quantification for the ε quantity in their definitions and discussions. This intervention by a facilitating researcher to bring together a problem and solution, and the resulting explicit resolution of the problem is represented in the lower right of Figure 3. Furthermore, as soon as their "acceptable range" was universally quantified, Megan and Belinda accordingly also referred to the "some point n" as changing depending on the particular size of the "acceptable range." This consisted of an implicit resolution to the problematic issue of the "some point n" being fixed, and is also represented in Figure 3.

We now return to the moment in the teaching experiment represented in Figure 1 and trace how the other two problematic issues first became explicit problems and how Megan and Belinda resolved them. The reader will recall that the students' third definition introduced the notion "at some point n".

Definition 3: As $n\to\infty$, at some n, a_{n+1} becomes closer to (or is) 5.

Unknowingly, the students had introduced ambiguity in a couple of important ways – first, in their definition "n" refers *both* to the static point on the n-axis beyond which the sequence behaves in a particular way and as the index for all of the individual terms of the sequence. Second, the phrase "a_{n+1} becomes closer" represented their attempt to articulate behavior of terms in the sequence after the "some n." Unfortunately, if n is viewed as fixed, so is n+1, and if n is viewed as general, then so must n+1. As the students' reinvention progressed during the second day, both of these ambiguities remained problematic issues, as can be seen in the following formulation.

Definition 20: A sequence converges to 5 as $n\to\infty$ only if there exists a value n after which the $|5-a_n|$ does not exceed your chosen error and $|5-a_{n+1}|<|5-a_n|$.

In an effort to turn problematic issues into actual problems for the students, the facilitating researchers at times served as *conflict producers*, asking questions designed to invoke cognitive conflict for the students. We illustrate such an instance in the following excerpt, wherein Craig directed Megan and Belinda's attention to the dual use of n in their definition.

Craig (aka, a Conflict Producer): My question was this $5-a_n$, what does a_n refer to?…The reason I'm asking is because it occurs to me that we're using n to mean, given some error, a static place here, right?
Megan & Belinda: Mmm-hmm
Craig: And then this n refers to a whole bunch of points.
Megan: Yeah.

Craig: Um, could it clear up confusion to call one of them something else maybe?
Belinda: Maybe a big N for this?

As she spoke, Belinda erased the n representing the static place on the n-axis, and replaced it with N (see Figure 4). Megan and Belinda then refined their definition to reflect this change.

Definition 21: A sequence converges to 5 as $n \to \infty$ only if there exists a value N after which the $|5-a_n|$ does not exceed your chosen error after N and $|5-a_{n+1}|<|5-a_n|$.

For the remainder of the teaching experiment, Megan and Belinda consistently used N to denote the point on the n-axis beyond which the sequence behaved in a particular way.

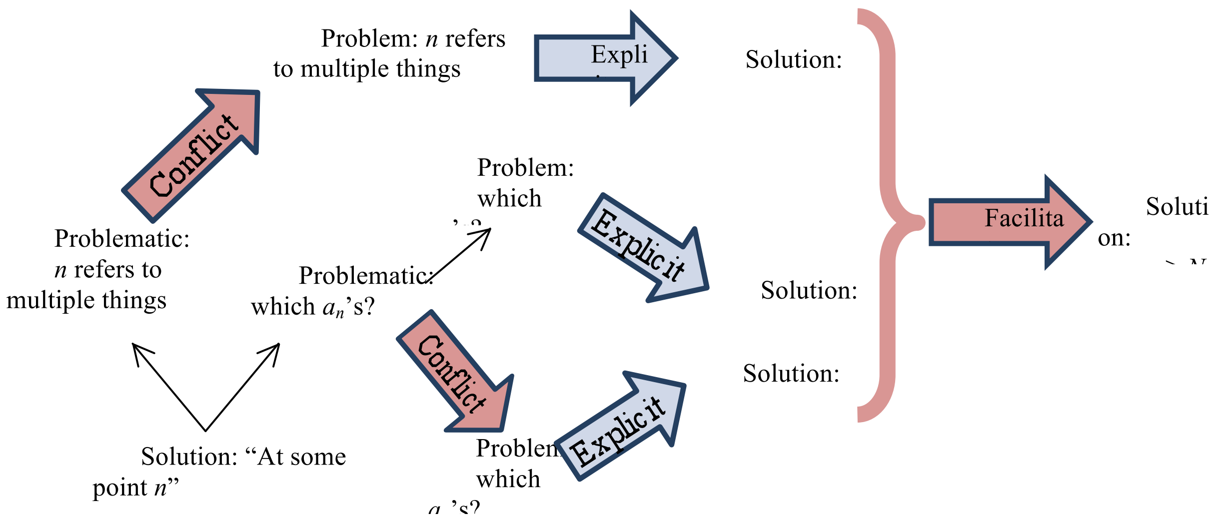

Figure 4. Students' and facilitators' activity in resolving problematic issues related to the index n.

The other ambiguity was resolved in a similar manner. In fact, each of the students (on two separate occasions) voiced concern about how to refer to the terms of the sequence after the "some point n." In this sense, the problematic issue had already become an actual problem for both Megan and Belinda (albeit on separate occasions). For instance, toward the end of the first session Megan refined Definition 12 by adding a subscript x to denote the terms of the sequence that were of interest to her, after which she verbalized the motivation for her refinement.

Definition 13: "A sequence converges to 5 as $n \to \infty$ i.f.f. $\exists$ n_x such that for all □□□ terms after $|5\text{-}a_n| \leq .01$"

After writing this, Megan explained, "I don't know. Just trying to think of a way to designate that point n forward and neglecting what's going on before that."

Similarly, during the second session, Belinda's attempt to capture the distinction between the terms of the sequence before and after the "some point n" suggests that this problematic issue had become an actual problem for her.

Craig (aka, a Conflict Producer): What does a_n refer to?
Megan: That's the approximation.
Belinda: That's whatever the term is. The approximation. All to the right of n.
Craig: Oh but not these guys back [here], because these are a_n's too aren't they?
Megan & Belinda: Yes.
Craig: But you're just saying the a_n's over here.

Megan: Mmm-hmm
Belinda: So maybe if we put like a little plus sign on that n to show that it's like, to the right.

As the third session progressed, Megan and Belinda still had not adopted a consistent means of denoting the terms of the sequence after N. To focus their attention on this issue, the facilitating researchers again acted as conflict producers.

Craig: [*pointing to* $|U–a_n|$ *in their definition*] You're saying that distance is within every error bound epsilon?
Belinda: Yeah.
Jason (aka, Conflict Producer): Is that true up there [*on the board*]?
Belinda: No, cause it's not true until you get to the N.
Megan: Yeah.
Belinda: So after some value N.

Next, Craig directed the students' attention to the graph seen in Figure 5.

Figure 5. Determining how to refer to sequence terms after N.

For the epsilon they had chosen, Craig asked the students to imagine that the corresponding N would occur when n=22. He then asked them to describe the relationship between the indices of the terms after that point and 22. They noted that the subsequent indices would all be larger than N, at which point Craig asked them to capture that relationship in writing. After some discussion, the following final definition emerged:

Definition 23: A sequence converges to U when $\forall \varepsilon > 0$ there exists some N, $\forall\ n \geq N$, $|U - a_n| < \varepsilon$.

Conclusions

In this paper, we reported results that support three general findings:
1. Ideas developed during the guided reinvention became useful to the students and persistent in their reasoning to the extent that they were generated as solutions to problems explicitly engaged and resolved by the students.
2. Aspects of approximation and error analysis introduced crucial conceptual entities and ways of reasoning for the students.
3. The researchers facilitating the guided reinvention process played important roles i) adhering to the cyclic process of definition refinement, ii) focusing students on

problems that needed to be resolved, and iii) at appropriate times, suggesting potential solutions to problems that remained intractable to the students.

The resolution of problems

Our distinction between problems and problematic issues emphasizes that it was the explicit solutions to problems consciously addressed problems that became generally useful to the students and persisted throughout their activity. Ideas that were raised in other contexts had neither of these properties. For example, students' initial universal quantification of epsilon occurred in a discussion separate from the problem of any single value of epsilon allowing non-examples. In this instance, although they fully understood the idea, they were not able to use it to advance their definition. Only when the universal quantification of epsilon was recognized as solving an explicit problem was it used and adopted. In general, powerful use of logical quantifiers and mathematical expressions emerged only after the students had i) fully developed the underlying conceptual structure of convergence in informal terms, ii) wrestled with the problem of how to rigorously express those ideas, and iii) seen the quantifiers and expressions as viable solutions to these problems.

The students' recognition and resolution of problems in their reinvention efforts were aided considerably by the presence of the examples they constructed at the start of the experiment. These examples served as sources of cognitive conflict when their definition failed to fully capture the necessary and sufficient conditions under which sequences converge. For example, the students' initial definitions were predictably couched in language that was vague, intuitive, and dynamic. The students immediately identified weaknesses in these definitions as they applied them to their examples that increase monotonically to 4, alternate around 5 or behave erratically before eventually looking like a standard example of a convergent sequence. Having identified these weaknesses, they also looked to their examples to provide direction for their revisions. This pattern of evaluating and refining their definitions against the examples repeated over the 23 cycles during the first three days of the teaching experiment.

Since the students' final definition was constructed through several iterations of a problem-solving process, they developed strong ownership over each piece and the overall organization. When shown a formal definition in a textbook, Megan and Belinda laughed and were delighted to see "their" definition in a book. The students expressed strong appreciation for the power of the quantifiers and mathematical notation in their definition, citing multiple problems that each part efficiently resolved and extolling its benefits over their earlier definitions stated in vague, informal language.

The Effect of an Approximation Scheme for Limits

The two students in this teaching experiment had only experienced instruction aimed at developing a systematic approximation scheme for reasoning about limits for a portion of one calculus course. Consequently, it is not surprising that they did not immediately invoke this scheme as they began to wrestle with generating a definition of sequence convergence and that the scheme emerged in pieces. Nevertheless, it did not take them long to turn to approximation ideas, and each portion of their evoked scheme emerged in response to particular problems for which it was well-suited to address. We note that these students progressed much more quickly towards a formal definition and through resolving several cognitive challenges than students not introduced to the approximation framework (Swinyard, in press). Once evoked, the students' ideas about approximation remained consistent. Further, their images and application of their scheme were sufficiently strong to provide them considerable guidance and conceptual support for reasoning about the formal definition.

The students' familiarity with a previously established approximation scheme provided significant leverage for i) focusing on relevant quantities in the formal definition, ii) fluently working with the relationships between these quantities, and iii) making the necessary but difficult cognitive shift to focus on N as a function of ε (Roh, 2008; Swinyard, in press). For example, during the first 12 minutes of the teaching experiment, the students did not invoke language about approximations to describe aspects of a sequence $\{a_n\}$. During this time they did not discuss or represent the quantity $|a_n - 5|$ in any form and all descriptions of convergence involved informal dynamic language. Twelve minutes into the interview, Megan rearticulated her requirement that the terms be getting closer to the limit in approximation language, "the error gets smaller and smaller at some point, after some value n [*holding her hand up and pinching her fingers together as she swept her hand to her right*]." When Craig asked what she meant by error, both students described the limit as the value being approximated, the terms a_n as the approximations, and the distance between the limit and the approximations as the error, which they represented as $|a_n - 5|$. These ideas and representations immediately appeared upon their invoking of approximation language and became an integral part of their arguments and written definitions. In the middle of the first session, they reformulated their idea that "each subsequent term is closer to 5 than the previous term" as the inequality "$|a_{n+1} - 5| < |a_n - 5|$" which they then used consistently throughout the remaining sessions.

After first introducing "approximations" and "errors," Megan and Belinda shifted to discussing how close the terms needed to get to 5 to consider the sequence convergent to 5. Twenty-six minutes into the session, as discussed in our results on the development of Definition 12, the students invoked the idea of an error bound (corresponding to ε in the formal definition) to address this question and focused on how to make the error smaller than this bound. At this point they seamlessly transferred their prior use of "some point n" (corresponding to N in the formal definition) to indicate the point at which approximations entered the desired error bound. Afterward, they consistently reasoned that this "point n depends [on] what the acceptable error is." For the remainder of Day 1 and throughout Days 2 and 3 of the teaching experiment, the students continued to rely on this approximation scheme to describe the relevant quantities and to keep track of the relationships among them.

The Nature of Co-Productive Activity of Students and Researchers in Guided Reinvention

We conclude our discussion with some observations of the important roles played by the researchers in guiding the students defining activity. The observing and facilitating researchers constantly monitored the students' progress, assessing the potential value of wrestling with the problems in which they were currently engaged, and being vigilant of the emergence of problematic issues not noticed by the students. Based on these assessments the facilitating researchers had three essential mechanisms for steering the discourse in a productive manner while complying with Gravemeijer's (1998) "basic principle of intellectual autonomy."

Most frequently and preferentially, the facilitators simply asked the students to return to or move to the logically next phase in the cyclic refinement process illustrated in Figure 1. For example, after the students had written a definition, we would typically simply read it back to the students and wait for any comments or clarification. If the students did not apply it to the examples on their own, we would ask them if it kept all examples in and non-examples out. This regularly led to recognition of conflicts between their stated definition and concept images, which they would begin to attempt to resolve. When they had generated potentially useful ideas, we would ask them to try to incorporate them into their definition, thus beginning a new cycle.

When guiding the general process was not sufficient to maintain productive activity, a slightly stronger intervention was required, but we strove to be careful in these moments to maintain the students' intellectual autonomy. As illustrated in the results, several problematic issues never surfaced explicitly through the students' own reflection, requiring our action as *conflict producers*. We only engaged in this role when we became convinced that the students were unlikely to recognize a particular issue that needed resolution. Often asking them to consider a specific portion of their definition in light of a specific example was sufficient to initiate the recognition of a conflict. On other occasions, students wrestled with a problem for a significant amount of time, and we became convinced that they would not resolve it on their own in a reasonable amount of time. In these instances, we played the role of *solution providers*. In order to maintain intellectual autonomy, we attempted to only a provide solution when we felt the students had a sufficient understanding of its elements that they were prepared to adopt immediate ownership of it and directly apply it to the problem at hand. An optimal example was bringing the explicit problem of any one epsilon value allowing non-examples into the same conversation with their previously expressed universal quantification of epsilon represents. That the students immediately agreed with the idea, applied it to several examples, incorporated it into their definition, and persisted with its use throughout the remainder of the sessions supports our decision that this would be a productive intervention.

Limitations and future work

This study drew from data collected in a teaching experiment with only two students and we acknowledge that each individual will follow unique paths. Further, orchestrating this type of discussion for an entire class would certainly involve significant differences from what was possible with focused attention on two students. Nevertheless, these students' reinvention of the definition serves not only as an existence proof that students can construct a coherent definition of sequence convergence, but also as an illustration of *how* students might reason as they do so. Our findings shed light on several relevant cognitive challenges engaged by the students, how they resolved these difficulties, and the resulting conceptual power derived from their solutions. These results are guiding our future work to develop, evaluate and refine classroom activities for introductory analysis courses. A further limitation of the study is that students' refinements of the definition were entirely based on checking against their concept image, thus was very descriptive in nature. As a result, we expect their definition to be useful for the students in powerfully and precisely describing their new image of convergence, but we do not expect them to be able to immediately use their definition to prove theorems. We are currently pursuing refinements to the teaching experiment protocols to incorporate assessments and justifications of statements about sequences to strengthen this aspect of students' reasoning.

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