RELATIONSHIPS BETWEEN QUANTITATIVE REASONING AND STUDENTS' PROBLEM SOLVING BEHAVIORS

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Over the past half-century, one strand of problem solving research has investigated problem solvers' behaviors during various phases of the problem solving process. A second strand of problem solving research has explored the mental actions, reasoning abilities, and understandings involved in using specific concepts to solve novel problems. These two strands of research provide valuable contributions to mathematics education, but they have remained as separate strands of inquiry within the research literature. This paper reports on the results of a study that investigated three students' problem solving behaviors in the context of their confronting novel problems when learning central ideas of trigonometry. The study's findings illustrate specific mental processes that enabled or hindered the students in engaging in productive problem solving behaviors. Particularly, the study describes how a student's ability to engage in quantitative reasoning shaped their movement through the problem solving phases.

Key Words: Precalculus, Problem Solving, Student Reasoning, Quantitative Reasoning

Introduction

Problem solving has been a focus of mathematicians and mathematics educators for the past half-century, with several studies characterizing students' and mathematicians' problem solving processes and performance (e.g., M. Carlson, 1999; M. P. Carlson & Bloom, 2005; DeFranco, 1996; Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007). These authors describe problem solving as a complex process of interrelated factors (e.g., affect, monitoring, and conceptual knowledge) and phases (e.g., planning and checking) with recent studies (M. Carlson, 1999; M. P. Carlson & Bloom, 2005; Schoenfeld, 2007) revealing the interaction of various factors in the problem solving process.

Another strand of mathematics education research investigates student thinking and reasoning in the context of students acquiring an understanding of a mathematical concept or idea as students encounter novel problems (e.g., Ellis, 2007; Hackenberg, 2010; Moore, 2010; Oehrtman, Carlson, & Thompson, 2008; Smith III & Thompson, 2008; Thompson, 1994a, 1994b). These studies' findings provide insights into the critical reasoning processes that support productive problem solving abilities. This collection of studies particularly identifies quantitative reasoning, as defined by Smith III and Thompson (2008), as critical for developing flexible and coherent understandings of mathematical topics.

Quantitative reasoning (Smith III & Thompson, 2008) refers to the mental actions involved in conceptualizing measurable attributes (quantities) of objects and relationships between these measurable attributes. In addition to the studies that highlight the role of quantitative reasoning in students acquiring understandings of a mathematical idea, Moore, Carlson, and Oehrtman (submitted) identified that quantitative reasoning is an essential reasoning pattern that leads to students building the mental imagery needed to construct meaningful formulas and graphs.

This study builds on the current body of problem solving research by investigating the reasoning students employ when moving through the problem solving phases. In order to accomplish this goal, I explored three precalculus students' thinking as they solved novel problems that required reasoning about angle measure and trigonometric functions. The students participated in a 5-week teaching experiment (Steffe & Thompson, 2000) that included a series of group teaching sessions. Each student also participated in a series of exploratory teaching interviews.

In order to describe the students' thinking during the problem solving process, I initially examined individual student written solutions and interview data. I then classified their mental actions in terms of the problem solving phases of orienting, planning, executing, and checking and drew relationships between their mental actions and their behaviors during the problem solving phases. The results of this analysis illustrate that the students' ability to engage in quantitative reasoning and build rich quantitative structures of a problem context influenced their actions and the products that they constructed during each of the problem solving phases. The study's results offer insights into the mental actions that either hinder or support students' engagement in productive problem solving behaviors.

Background

In starting any discussion on problem solving, it is natural to begin by referencing Pólya's (1957) initial work in describing four problem solving phases: a) understanding the problem, b) developing a plan, c) carrying out this developed plan, and d) looking back at the solution. Since Pólya's initial work, contributions to the body of literature on problem solving (see Lester Jr. (1994) and Schoenfeld (2007) for exhaustive reviews) provide many insights into the characteristics of successful problem solvers, including comparisons of the behaviors exhibited by successful and unsuccessful problem solvers. These studies have revealed many factors that contribute to an individual's problem solving ability, including an individual's knowledge and her/his attitudes and beliefs about mathematics and problem solving. Several authors illustrate (M. Carlson, 1999; DeBellis & Goldin, 1999; Hannula, 1999) the influence of affective variables (e.g., beliefs, attitudes, and emotions) on an individual's problem solving activity, while other studies highlight the critical role of metacognition in problem solving (Schoenfeld, 1992; DeFranco, 1996, Carlson, 1999). As a particular example, Schoenfeld (1992) identified that successful problem solvers often monitored and regulated their own actions while unsuccessful problem solvers did this on a less frequent basis.

In the early 1990s, the main focus of research on problem solving turned away from exploring problem solvers' behaviors (Schoenfeld, 2007). Rather than focusing on the mental actions of problem solvers and attempting to determine how and why problem solvers make their choices, the emphasis of the community turned to exploring learning environments for problem solving. These studies explored sense-making classrooms, discourse communities, participant accountability, and productive classroom cultures. Although these studies provide valuable insights into social environments, I note that the main focus of problem solving transitioned away from investigating students' cognitive actions to looking at social settings, though recent studies (M. P. Carlson & Bloom, 2005; Schoenfeld, 2007) suggest that much is to be learned by continuing to examine the cognitive actions involved in problem solving. These insights are needed to better understand how to foster the development of flexible and productive problem solving abilities in students.

In an attempt to provide a finer characterization of problem solvers' cognitive processes, Carlson and Bloom (2005) investigated the problem solving activity of 12 mathematicians. Drawing from analysis of interviews with the mathematicians, as well as previous research on problem solving (DeBellis & Goldin, 1999; DeFranco, 1996; Hannula, 1999; Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007), the authors created the *Multidimensional Problem-Solving Framework*. This framework expands on Póyla's problem solving phases by characterizing how various attributes (e.g., affect and the use of heuristics) influence a problem solver's behaviors during the problem solving phases. The authors also identified multiple problem solving cycles within the four problem solving phases (e.g., orientation, planning, executing, and checking). For instance, the authors noted that the mathematicians frequently engaged in a *conjecture-imagine-evaluate* sub-cycle of the planning phase to mentally play out a solution. The authors also identified that the mathematicians exhibited problem solving behaviors indicating a more formal and deliberate *plan-execute-check* problem solving cycle.

Carlson and Bloom's (2005) study of mathematicians' problem solving behaviors emphasized that further explorations are needed to identify the role of various reasoning patterns in problem solving. Particularly, the authors claimed, "[the mathematicians'] ability to play out possible solution paths to explore the viability of different approaches appears to have contributed significantly to their…problem solving success," while suggesting that subsequent studies explore such connections between problem solvers' thinking and their problem solving behaviors (M. P. Carlson & Bloom, 2005, p. 69). In response to the authors' call for studies that explore the mental actions involved in problem solving, myself and two colleagues (Moore, Carlson, & Oehrtman, submitted) examined the role of quantitative reasoning (Smith III & Thompson, 2008) when precalculus students orient to novel problems. We were curious to know why precalculus students were exhibiting difficulties that were not reported in studies of expert problem solvers, and leveraged Smith III and Thompson's (2008) model of quantitative reasoning to explain these difficulties.

In short, quantitative reasoning (Smith III & Thompson, 2008) refers to the mental actions involved in conceiving of situation such that it is composed of measurable attributes, called quantities, and relationships between these quantities. The quantitative structure that results from these mental actions can support students in developing meaningful formulas, calculations, and graphs. In analyzing precalculus students' problem solving behaviors, we (Moore, Carlson, & Oehrtman, submitted) identified the critical role of quantitative reasoning during the orientating problem solving phase. The students' solutions to the problems relied on the images they constructed of the quantities and their relationships when initially making sense of the words in the problem. Students who developed robust images of a problem's context constructed quantitative structures that supported their determining meaningful formulas and graphs. In light of this finding, the authors concluded that problem solving research could benefit from exploring further connections between quantitative reasoning and students' problem solving behaviors.

The argument for investigating connections between quantitative reasoning and problem solving is further supported by findings (Ellis, 2007; Hackenberg, 2010; Moore, 2010; Oehrtman et al., 2008; Smith III & Thompson, 2008; Thompson, 1994a, 1994b) that describe quantitative reasoning as an important aspect of learning mathematics. These studies explored students' thinking as they solved novel problems with the intentions of characterizing the students' thinking relative to a particular mathematical topic. Though these studies did not explicitly explore the students' solutions in the context of literature on problem solving (e.g., the problem solving phases), the studies' findings imply that quantitative reasoning is a critical component of solving novel problems. In light of the collection of research findings on quantitative reasoning, I conjectured that quantitative reasoning provides a useful lens for exploring students' problem solving behaviors during the various problem solving phases.

Methods and Subjects

Three students from an undergraduate precalculus course at a large public university in the southwest United States participated in the study. The researcher (myself) was the instructor of the course. I chose the participants on a voluntary basis and they were monetarily compensated for their involvement. I drew the participants from a precalculus classroom that was part of a design research study. Theory on the processes of covariational reasoning and select literature about mathematical discourse and problem solving (M. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; M. P. Carlson & Bloom, 2005; Clark, Moore, & Carlson, 2008) informed the design of the course. All three participants (Judy, Zac, and Amy) were full-time students at the time of the study. Judy was a female in her mid-twenties and a biochemistry student. Zac was a male in his early twenties and an ethnomusicology and audio technology major. Amy was a female in her late teens and an undeclared major planning on pursuing nursing.

The design of the study followed a teaching experiment methodology, as described by Steffe and Thompson (2000). The teaching experiment consisted of eight 75-minute teaching sessions within a span of five weeks. Each teaching session included all three participants, the instructor/researcher (myself), and an observer. Immediately after each session, I debriefed with the observer to discuss various observations during the teaching sessions and the design of future teaching sessions and interviews. I also conducted exploratory teaching interviews with each participant. I used the exploratory teaching interviews to gain additional insights into the participants' thinking. These one-on-one interviews followed the teaching experiment principles, and helped me to identify the students' thinking and the influence of their thinking on their behaviors during the problem solving phases.

I analyzed the data using an open and axial coding approach (Strauss & Corbin, 1998). I first analyzed the students' behaviors in an attempt to determine the mental actions that contributed to their solutions. I then characterized the mental actions inferred from the students' behaviors in terms of the problem solving phases identified by Carlson and Bloom (2005). This phase of the data analysis involved identifying how the students' mental actions influenced their behaviors during the four problem solving phases. This approach to analyzing the data enabled me to classify how various reasoning patterns related to the students' problem solving behaviors. Lastly, I compared and contrasted the students' mental actions and problem solving behaviors. In this stage of the analysis, I drew connections between the students' propensity to engage in quantitative reasoning and their activity during the problem solving phases.

Results

This section describes the students' solutions to novel problems by characterizing their mental actions in the context of the problem solving phases of *orienting, planning, checking, and executing*. As the study progressed, Judy and Zac revealed similar approaches to solving novel problems, while Amy's problem solving behaviors differed considerably from Judy's and Zac's behaviors. Paralleling this observation, Judy's and Zac's mental actions were in stark contrast to Amy's mental actions when confronting the problems. In order to illustrate relationships between the students' thinking and their behaviors during the problem solving phases, I compare and contrast Zac's and Amy's solutions.

After an introduction to trigonometric functions in a circular motion context, the students attempted a problem set within a right triangle context (Table 1). Zac *oriented* to the problem by constructing a diagram of the situation and labeling the given values (Figure 1).

Table 1
The Empire State Building Problem

While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

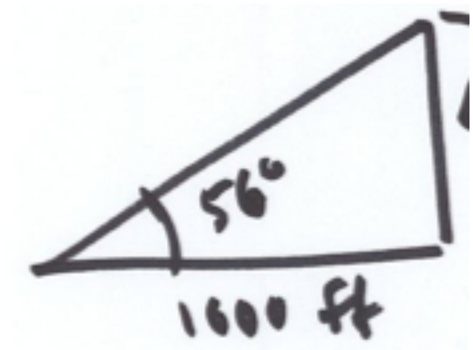

Figure 1. Zac's initial diagram on The Empire State Building Problem.

After drawing a diagram of the situation, Zac explained, "From the circle, or triangle, we can determine that cosine of fifty six degrees is equal to one thousand feet (*writing corresponding equation*)…one thousand feet is equal to [a number of radius lengths], because cosine fifty six degrees is determined in radius lengths." After converting the given angle measure to a number of radians, Zac continued to explain his solution (Excerpt 1).

Excerpt 1

Zac: Ok, alright, so. Scratch that, point nine eight. This (*referring to cosine expression*) is equal to a thousand feet…then a thousand feet is equal to the radians times the radius length (*writing '(rad)(r)'*), or *r*.

KM: Ok.

Zac: Ok, 'cause the radians is just a percentage of the radius length. Oook. So, now what I want to do is figure out what cosine point nine eight is actually equal to, and using that I can find out what the radius length is (*pointing to r*). So then when I do sine of point nine eight, I already know what the radius length is, so when I get the answer to that (*referring to sine*) all I have to do is multiply by the radius length and I'll get that part (*identifying the vertical segment on his triangle*).

KM: Ok, so if you wanna go ahead and do that.

Zac: Ya. Ok, so, (*using calculator*) cosine point nine eight is equal to, (*writing*) equals point five six radians. And so, all I have to do is (*using calculator*) divide one-thousand by point five six, as shown in this equation right here (*referring to* '*1000=(rad)(r)*'), to isolate the radius all I have to do is divide it by the radians. (*using calculator*) And I get a big number. So that means (*writing*) *r* is equal to one seven eight five point seven one. So then I do (*writing*) sine point nine eight, (*using calculator*) and I'm given a radius length, or a percentage of a radius length, (*writing*) equal to point eight three. Now all I have to do is multiply that (*writing*) by *r* and I'll get the length of that side (*pointing to vertical segment on his triangle*), so, times (*using calculator*) one seven eight five point seven one. So that means the length of that side is equal to (*writing*) one four eight three point O three feet. Figure definitely not drawn to scale.

Though Zac constructed (Figure 2) a mathematically incorrect equation ($\cos(0.98) = 1000$), he *planned* his solution by mentally playing out a sequence of calculations. Specifically, Zac anticipated "what cosine point nine eight is actually equal to," and explained that $\cos(0.98)$ represented a fraction of the radius without *executing* a calculation to determine this value (lines 5-11). Instead, he imagined determining a measure in radii, using this measure to determine the radius length, and then determining a value that represented a multiplicative relationship between a length and the radius (e.g., the output of the sine function).

Zac's actions reveal him *planning* a sequence of calculations without immediately *executing* the calculations (lines 8-11). Then, as Zac *executed* the calculations, he continually referred to the quantities of the situation and the meaning of the various measurements he obtained to monitor and *check* his solution (lines 14-27).

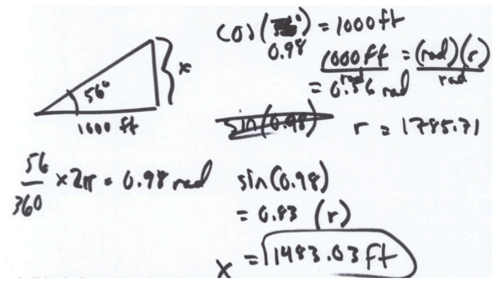

Figure 2. Zac's work on The Empire State Building Problem.

A right triangle context was not introduced during a teaching experiment sessions previous to this problem, yet Zac reasoned about multiplicative relationships between a radius and various lengths to construct a quantitative structure that enabled him to *plan* his solution before *executing* calculations. After providing his solution, Zac also explained that he *oriented* to the problem by conceiving of the hypotenuse of the right triangle as the radius of a circle (Figure 3). This *orienting* act, which was unobservable to myself previous to his drawing Figure 3, aided him in constructing a robust image of the problem's context (e.g., a well-developed quantitative structure) that supported him in *planning* his solution.

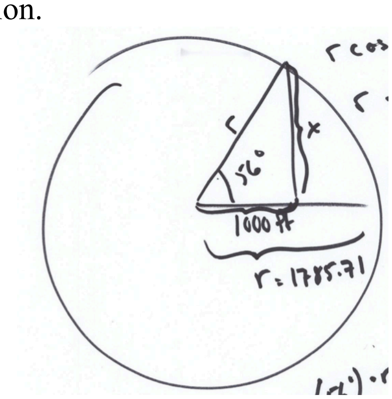

Figure 3. Zac's refined diagram on The Empire State Building Problem.

As an example of another student's solution, Amy read the problem statement, drew a diagram, and then claimed, "I know how to do this, I did this last night…I just did this…give me the first step and maybe I can figure it out from there…and we just learned this." Her claim, "I know how to do this…give me the first step," implies that she expected to recall a post procedure when solving the problem.

Amy subsequently suggested converting the angle measure to a number of degrees (the problem differed from that in Table 1 and included a given height of 670 feet and a given angle measure of 1.1 radians), but expressed that she unsure how the measure related to the goal of the problem. Amy then attempted to determine the hypotenuse of the right triangle (Excerpt 2).

Excerpt 2

Amy: Um (*pause*). Doesn't six hundred and seventy divided by this side (*pointing to the side adjacent the given angle*) give us the radius? Right?
KM: What do you think?
Amy: Well I know it does, but.
KM: Six hundred and seventy divided by what side?
Amy: Right here (*pointing to the side adjacent the given angle*).
KM: So why do you think that gives us the radius?
Amy: I know it does.
KM: So why do you know it does?
Amy: 'Cause that's what is in my notes.

Amy attempted to recall a calculation (lines 1-2), looked to the researcher for approval, and claimed that her suggested calculation was in her notes (lines 8-10). In this case, Amy's conjectures focused on calculations not rooted in quantitative relationships, which likely limited her ability to *check* the correctness of her conjectures beyond referencing her notes. Amy was then unable to move forward on the problem.

Over the course of the study, Zac and Amy exhibited actions similar to those described in Excerpts 1 and 2. After reading the problem statement and drawing a diagram, Amy frequently *oriented* and *planned* her solution by attempting to recall past problems, calculations, and procedures. Amy's explanations also revealed that the recalled calculations and procedures did not stem from quantitative relationships. For instance, Amy often immediately *executed* (or emphasized a need to *execute*) recalled calculations (and formulas), but she was unable to justify or *check* these calculations beyond claiming the she used the same procedure or calculation to solve a previous problem. As a result, she frequently had difficulty obtaining solutions to the posed problems.

Zac (and Judy), on the other hand, frequently *oriented* to a novel problem by drawing a diagram and they then *planned* their solution by discussing quantitative relationships and calculations that stemmed from these relationships. Such actions enabled Zac and Judy to anticipate *executing* a sequence of calculations without needing to perform the calculations. As Zac and Judy *executed* calculations, they also maintained a focus on the context of the problem in order to monitor and *check* their progress.

Due to Amy experiencing difficulty in providing a solution to the problems, I implemented tasks that were more conducive to recalling calculations or procedures (e.g., tasks that required only one or two calculations to correctly solve the problem). I compared and contrasted the students' actions during these problems and I also compared the students' solutions to the solutions they provided during more complex problems (e.g., Table 1).

The Arc Length Problem (Table 2) prompted the students to determine various arc lengths when given an angle measure in degrees.

Table 2
The Arc Length Problem

Given that the following angle measurement θ is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Task includes diagram with an angle and three concentric circles centered at the vertex of the angle*)

After reading the problem statement and identifying each given measurement on a diagram (*orienting*) Zac *planned* a sequence of calculations previous to *executing* the calculations (Excerpt 3).

Excerpt 3

1	Zac:	Um, ok. So what I plan on doing for this one is converting thirty-five
2		degrees into radians. And a very easy way of doing that is putting thirty-
3		five over three sixty is equal to x over two pi (*writing corresponding*
4		*equation*)…And then with that all I have do is just multiply the answer
5		(*pointing to x*) by two inches, two point four inches, and two point nine
6		inches (*pointing to each value in the problem statement*) to get the
7		different arc lengths (*identifying each arc length with his pen tip*) right
8		there, because radians is just a percentage of a radius.

Zac first described converting the angle measure from degrees to radians without *executing* the conversion (lines 1-2). He then reasoned about a multiplicative relationship between a subtended arc and the radius (a percentage of the radius) in order to anticipate *executing* a calculation that determines each arc length (lines 4-8). These actions illustrate Zac reasoning about a measurement in radians as a multiplicative comparison (e.g., a quantitative relationship) between an arc length and the radius length to *plan* his solution, which supported him in mentally playing out a sequence of calculations without experiencing a need to *execute* each calculation previous to considering the subsequent calculation.

After Excerpt 3 and in order to describe his reasoning when converting the give angle measure to a number of radians, Zac explained, "Well what you're doing is just technically finding a percentage. Like thirty-five over three sixty is (*using calculator*), is nine point seven percent of the full circumference." Similar to Zac's ability to reason about a multiplicative relationship between a subtended arc and the radius to anticipate determining an arc length, Zac's explanation reveals that his ability to reason about angle measure as conveying a fraction of a circle's circumference subtended by the angle supported him in reasoning about a converted angle measure without needing to *execute* the calculation. Thus, Zac's constructed quantitative structure enabled him to *plan* his solution by mentally playing out a sequence of calculations.

As an example of Amy's solution on The Arc Length Problem (Table 2), consider her initial attempt to solve the problem (Excerpt 4).

Excerpt 4

1	Amy:	I remember doing this, I'm just trying to remember how. Ok. Radius.
2		Hmm, and the angle measure's thirty-five degrees. And I'm trying to
3		determine the length (*tracing arc length*), like in inches?
4	KM:	Ya.
5	Amy:	Ok. Alright, let's see. Um. Two inches. And it's thirty-five degrees out of
6		three hundred sixty degrees (*writing ratio of 35 to 360*). Hmmm. If two
7		inches (*pause*), no. Hmm, I swear I know how to do this, I just can't
8		remember. (*long pause*) (*sigh*) Ok. (*long pause*) Hmm. I'm trying to
9		determine the length of the arc. Oh I promise I know how to do this. I
10		can't remember though (*long pause*).

After reading the problem statement, Amy attempted to "remember" a solution to the problem (line 1), though she did not solve a similar problem during a previous session. Amy continually emphasized that she knew how to complete the problem, but that she was unable to recall a solution for the problem (lines 5-10). Amy's utterances suggest that she equated solving the problem to implementing a previously determined procedure, and though she did write a ratio, she did not provide a meaning for this ratio (lines 5-6).

Amy subsequently suggested using the equation $\frac{35}{360} = \frac{x}{2\pi}$, but explained, "I don't know if, 'cause then why would I, I don't know. I think I need to incorporate the radius somehow." I then asked Amy to explain her equation and she suggested an uncertainty about the result of solving the equation. After *executing* a cross-multiplication procedure to solve for x, she explained that the value was a number of radians and admitted to memorizing the equation during a past session. After determining the angle measure in radians, I prompted her to continue her attempt to solve the problem (Excerpt 5).

Excerpt 5

1	Amy:	Ya, wait. Do I take the percentage? No. (*looking at the researcher*) Is it
2		the percentage divided by the radius or opposite? No, that, no. That's not
3		right (*pause*). That's sixty one percent of one radian. And (*pause*). My
4		radius is two. Would using cosine and sine be (*pause*) helpful?

Similar to Excerpt 2, Amy exhibited *planning* actions in the form of suggesting calculations and "using cosine and sine," which were introduced in a previous session. She also looked to the researcher for approval of her suggestions and she subsequently claimed that she was unable to proceed.

Amy often exhibited actions consistent with those in Excerpts 2, 4, and 5. She predominately attempted to recall calculations and procedures after reading a problem statement and regularly *executed* calculations or expressed a need to *execute* a calculation without first considering subsequent steps in her solution or explaining a meaning for these calculations. As another example of Amy's actions, consider her response when determining a circle's circumference given that an angle subtended 0.3 inches, or 22%, of a circle's circumference. Amy obtained a correct solution to the problem by determining an equation that enabled her to *execute* a cross-multiplying procedure (Excerpt 6).

Excerpt 6

Amy: Given that an arc-length of point zero three inches is twenty two percent of a circle's circumference, what is the circle's circumference? Alright, so I've got an arc length that is point zero three inches. A circle's circumference is pi times diameter. Isn't that pi times the diameter?

KM: Pi times the diameter.

Amy: (*writing* πd) And then, ok, all we have it point three. So we've got point three and the whole circle is pi times the diameter (*writing .03 above* πd). Of the circle's circumference. Hmmm. This would be better if I had a radius. Ok, it's twenty-two percent of the circumference. Twenty-two is the result of one hundred (*writing the ratio of 22 to 100*). (*calculating 100 times .03*) I don't know if this is right, I'm just gonna give it a shot. I get three. Oh, I need an x somewhere. Hmm.

KM: So what'd you do there? You did...

Amy: Ya, well I was gonna try and like cross-multiply and everything. But I...

KM: So what do you mean by you need an x? What are you referring to?

Amy: I need something that I'm gonna solve. Which would be the rest, which would be the whole circumference. So, ya, it would be like point three over x if we wanted to find it in inches (*replacing* πd *with* x). So it would be equals 3 (*writing* $22x=3$) and that doesn't make sense.

KM: So what doesn't...

Amy: The whole circle's point one three inches. I mean, the circumference. I don't know if that makes sense.

KM: So what do you think? How long was your arc length?

Amy: My arc length was point zero three, I just don't like uneven stuff.

After reading the problem statement, she recalled a formula for the circumference of the circle and desired to use this formula and the length of the radius to *execute* a calculation (lines 3-4 and 8-9). Amy subsequently constructed an equation between two ratios using a part to whole correspondence (lines 6-10), and though she was unsure of her equation she immediately attempted to *execute* a sequence of calculations (e.g., cross-multiplication) (lines 11-12). However, she was uncomfortable because the equation did not include "an x…something I'm gonna solve." After altering her equation to appease her discomfort and solving the equation, Amy encountered difficulty *checking* her solution beyond considering the aesthetics of the result (e.g., "uneven stuff").

Beyond using a part to whole correspondence when constructing an equation, Amy's actions did not suggest that her solution and calculations stemmed from a constructed quantitative structure. Rather, her solution stemmed from her deeming the problem to be appropriate for applying a cross-multiplication procedure. Her *planning* actions consisted of setting up an equation that supported *executing* cross-multiplication. Furthermore, without a well-developed quantitative structure to support her solution (e.g., the ratios Amy determined did not reflect a multiplicative relationship), Amy was unable to *check* her solution beyond trusting the memorized procedure and judging the aesthetics of the solution.

After this interaction, Amy was unable to explain her solution beyond claiming that cross-multiplication was a prescribed procedure for such a problem. She also added, "I like doing cross-multiplying," conveying that she experienced comfort in *executing* this procedure. On similar problems, Amy attempted to determine equations that supported *executing* cross-multiplication. When asked to consider an alternate solution in such cases, Amy often provided responses along the lines of, "I know there's an easier way, I just, never bothered to learn it. I like my method of cross-multiplying."

Discussion

The students' solutions imply a strong correlation between the nature of their mental actions and their corresponding problem solving behaviors during the various problem solving phases. Judy and Zac made sense of a problem's context by developing rich quantitative structures that they leveraged during the problem solving phases. Contrary to this, Amy focused on executing procedures and calculations and rarely conceptualized a problem's quantities, making little progress in advancing her solutions.

When orienting to a problem, Judy and Zac regularly drew and labeled a diagram of the problem's context with known and unknown measurements. They also discussed various relationships between quantities when drawing these diagrams. During the planning phase, they reasoned about relationships between quantities in order to *anticipate* executing a sequence of calculations (Excerpts 1 and 3). The students' ability to reason about relationships between quantities without performing numerical calculations enabled the students to engage in the *conjecture-imagine-evaluate* cycle identified by Carlson and Bloom (2005) in order to mentally play out their solutions. Also, in the case that Judy and Zac recalled formulas during the planning phase, they described these formulas in terms of the quantities of the situation and used the formulas to represent values without first evaluating the formulas.

When Judy and Zac executed their planned calculations, they continued to describe the calculations in terms of the quantities of the situation and consistently used a diagram to illustrate the quantity referenced by a determined value (Excerpt 1). Also, stemming from their robust quantitative structures that they constructed during the planning phases, the students established a quantitative meaning for the result of a calculation previous to performing the calculation. When executing calculations, Judy's and Zac's images of the problem contexts also supported their monitoring and checking the appropriateness of the calculated values. When they obtained values that were not consistent with their image of a problem's context, they returned to the context to further orient to the problem, check their solution, and modify their solution. Thus, Judy's and Zac's actions of constructing a robust image of the problems' contexts supported their productively engaging in the *plan-execute-check* cycle identified by Carlson and Bloom (2005).

Contrary to Judy's and Zac's behaviors, Amy exhibited an inclination to reason about calculations and procedures. When orienting to a problem, she often drew a diagram, but she infrequently labeled unknown values on the diagram and spent limited time verbally discussing a problem's context. Instead, she regularly referred to previously completed problems deemed similar to the current problem and attempted to recall previous procedures and calculations (Excerpts 2, 4, and 5). In the cases that she recalled previous procedures, she immediately executed the procedures without explaining or analyzing the procedure in terms of the problem's context. In the cases that she did not recall a previous solution, she often suggested calculations and encountered difficulty in determining a correct solution (Excerpts 2, 4, and 5). Furthermore, when I asked Amy to explain a meaning of her suggested calculations, she first executed the calculation and experienced difficulty determining how the obtained value related to the problem's context and the goal of the problem.

Amy's inclination to use calculations devoid of quantitative meaning also presented her difficulties when she attempted to check her solutions or calculations. Amy often relied on the aesthetic quality of her answers (e.g., values not "too big," "too small," or "uneven") and her comfort with the applied procedure (Excerpt 6). For instance, Amy felt confident when she encountered a problem that she could apply a cross-multiplication procedure. In the cases that the she believed her solution was incorrect or that she was unable to determine a solution, she looked to me for assistance. Amy's actions suggest that she did not construct quantitative structures that supported her solutions or engaging in the problem solving sub-cycles identified by Carlson and Bloom (2005).

Table 3 provides a summary of the students' problem solving behaviors while comparing Judy's and Zac's actions with Amy's actions during each problem solving phase. Overall, Judy's and Zac's actions reveal a continued focus on the context of the problem. Namely, Judy and Zac constructed quantitative structures that enabled them to engaged in connected and productive problem solving phases during both types of problems (e.g., the multi-step problems and the one- or two-step problems). Amy's actions during each phase, regardless of the problem type, centered on recalling and performing procedures and calculations, which did not support productive and connected problem solving phases. For instance, Judy and Zac frequently engaged in the *conjecture-imagine-verify* sub-cycle during the planning phase, which also supported their subsequent actions during the executing and checking phases. However, Amy did not exhibit actions that indicated she engaged in the *conjecture-imagine-verify* sub-cycle and a majority of her efforts were given to recalling a procedure to execute. These contrasting approaches to solving novel problems provide insights into the influence of a student's reasoning on her/his problem solving behaviors.

Table 3
Problem Solving Activity and Orientations

Phase	**Actions**	
	Judy and Zac	Amy
Orientation	- Spending a significant amount of time describing the context of the problem and developing an image of the problem's context (e.g., drawing a diagram of the situation). - Returning to the diagram of the situation to label determined values during their solution.	- Spending minimal time describing the context of the situation and drawing a diagram of the situation. - Rarely returning to the diagram of the situation after the initial orientation process - Recalling a previously completed problem deemed similar to the current problem.

Planning	- Reasoning about relationships between quantities to anticipate a series of calculations and obtaining unknown values previous to executing the calculations. - Recalling a formula and describing it in terms of quantitative relationships. - Continuing to reference the context of the problem and making alterations to the diagram of the context. - Engaging in the *conjecture-imagine-verify* sub-cycle to mentally play out a sequence of calculations.	- Recalling a formula and expressing a need to evaluate the formula. - Attempting to remember the steps and calculations of a previous procedure. - Expressing the need to execute a calculation previous to explaining a meaning of a calculation or considering subsequent calculations. - Looking to the researcher for suggestions.
Executing	- Describing calculations in terms of the quantities of the situation and relationships between these quantities. - Describing a meaning of a value previous to obtaining the value.	- Describing calculations in terms of numerical operations and numbers. - Executing calculations without referencing the context of the problem. - Encountering the need to determine a meaning of a calculated value. - Experiencing difficulty in determining a meaning of a calculated value and how the value related to the goal of the problem.
Checking	- Returning to the context of the situation when obtaining an unexpected value (e.g., returning to the orientation and planning phases). - Limited checking of calculations, as calculations were planned in terms of quantitative relationships. - Exhibiting confidence in their solutions.	- Checking numerical operations relative to a recalled procedure. - Focusing on the aesthetic quality of the solution (e.g., values not "too big" or "too small"). - Limited referencing to the context of the situation. - Looking for outside approval (e.g., interviewer or teacher) of calculations and results.

Conclusions and Implications

The students' problem solving approaches reveal relationships between students' reasoning and their problem solving behaviors. The students' solutions also reveal that their approaches to problem solving can either support or hinder their ability to solve novel problems. The study's findings suggest that students who construct robust images of a problem's context (e.g., a cognitive structure of quantities and quantitative relationships) are better able to engage in connected and productive problem solving behaviors that support their constructing logical arguments and solutions. Furthermore, the students' actions suggest that students can more flexibly recall and leverage understandings rooted in quantitative relationships to solve novel problems, as opposed to understandings that are rooted in memorized procedures and calculations. The students' actions imply that this claim holds for one- or two-step problems, in addition to more complex problems (e.g., problems that require fluent function conceptions).

The students' solutions to the problems also suggest that a student's ability and inclination to engage in quantitative reasoning influences their ability to exercise monitoring behaviors during the problem solving process. Judy and Zac were better able to monitor their actions, while Amy was often unable to judge the effectiveness and appropriateness of her solutions. These findings imply that understandings and reasoning rooted in quantitative relationships support students in not only developing resources of greater utility, but also improving their ability to monitor and analyze their thought process when solving novel problems. Several authors (Carlson and Bloom, 2005; Schoenfeld, 1992) highlight the critical role of these metacognition activities, and the findings of this study shed further light on the mental actions that engender metacognition behaviors.

In addition to contributing to the body of research on problem solving, this study's findings should inform curriculum designers and teachers about the reasoning abilities and problem solving behaviors that support students in solving novel problems. Instruction that focuses on performing calculations and procedures devoid of quantitative meaning may not support students in developing powerful problem solving abilities or knowledge structures that support solving novel problems. Rather, students are better prepared to solve novel problems when they engage in reasoning patterns and construct understandings that are rooted in quantities and quantitative relationships. For instance, Judy and Zac *did* suggest calculations and attempt to recall procedures at various (though rare) times during the study. In the case of executing a calculation that they could not anticipate, they often executed the calculations and then immediately attempted to interpret the resulting value in terms of a quantitative relationship with the values used to perform the calculation. By maintaining this quantitative focus, Judy and Zac were able to use such calculations to inform their solution. However, in the case that they attempted to recall a procedure or calculation without considering the quantities of the situation, they frequently failed to progress on the problem until they abandoned such a solution attempt.

Future Research

This study consisted of a teaching experiment in trigonometry and the students' problem solving behaviors were only investigated in the context of trigonometry problems. Still, the findings from this study (e.g., Table 3) can provide a foundation for future problem solving research that explores students' solutions to non-trigonometry problems. For instance, future studies that investigate students' problem solving behaviors within other mathematical topics can provide data to compare to and extend the results of this study.

The findings I present above suggest that students benefit from a quantitative approach to problem solving. Future research should investigate this claim and attempt to determine ways to support students in developing meaningful approaches to solving novel problems. During this study, I observed students alternating problem solving approaches between problems, as well as within a single problem. Studies should explore reasons for such transitions, and the instruction necessary to promote students developing problem solving approaches and reasoning patterns that support their constructing meaningful and correct solutions to novel problems. For instance, the findings from this study imply that understanding rooted in quantitative relationships support a student's ability to solve novel problems. This implication provides a path of exploration for future studies.

Judy and Zac also appeared to be more reflective (e.g., engage in monitoring and checking actions) during their problem solving activity. This may have been a result of their constructing quantitative structures that created foundations for reflective actions. Future research should investigate this phenomenon, and its implications for using problem solving to support students in learning mathematics.

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