MATHEMATICIANS' PEDAGOGICAL THOUGHTS AND PRACTICES IN PROOF PRESENTATION

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This descriptive study investigates the ways in which proofs are presented in upper division proof-based undergraduate mathematics courses at a large comprehensive research university in the Midwest. To pursue this inquiry, four faculty members who were teaching such courses were interviewed and three of the four participated in video-taped observations periodically throughout the course of the semester. Interview data were used to construct a framework with which to analyze the observation data. The observation data were analyzed to determine the level of engagement that the professor seemed to expect of the students, and to identify some proof presentation strategies that were used.

Keywords: proof presentation, teaching proof, expected engagement, traditional lecture

Introduction and Literature Review

The teaching and learning of mathematical proof has proven to be problematic for decades (Grassl & Mingus, 2007; Larsen, 2009; Larsen & Zandieh, 2008; Selden & Selden, 2003; Tall 1997). Research in this area usually comes in two flavors: investigating student thinking (Knuth, 2002; Larsen, 2009; Simpson & Stehlikova, 2006; Healey & Hoyles, 2000; Selden & Selden, 2003) and developmental research projects that focus on developing innovative ways to teach proof (Leron and Dubinski,1992; Larsen, 2009; Weber, 2006). These studies shed light on teaching and learning in the context of mathematical proof, but it is often difficult to apply these findings directly. A foundation of knowledge about the range of current teaching practices is needed before we can adequately understand how to improve student learning.

It is generally acknowledged that lecture is the norm in most university mathematics classrooms. The traditional lecture style has been criticized by many, especially by those who propose alternative, more interactive teaching methods such as computer activities or guided reinvention of the concepts (Leron & Dubinski, 1995; Larsen, 2009). These types of reform curriculum require a dramatic shift in the delivery of the material from teacher-centered to student-centered. Leron (1985) merely called for a divergence from a linear proof presentation method in favor of "heuristic" presentations, which give the audience a better idea of how the ideas were constructed. In the traditional lecture format, students are expected to learn how to prove theorems by attending to the presentations of proofs given by their instructor in class and by working the homework problems that are assigned. Even though the presentation of proof in class is a primary tool for teaching proof, there are few research studies addressing this topic (Weber, 2010).

We believe that the traditional lecture style of teaching mathematical proof should be investigated more closely in order to catalog the strategies that professors are using. In a recent article, Speer, Smith, & Horvath (2010) claim that "very little empirical research has yet described and analyzed the practices of teachers of mathematics" (p. 99). They mention that although the "effects of instructional activities have been examined… the actions of the teachers using those activities have not" (p. 101). They conducted an extensive literature search for published articles addressing collegiate teaching practice, only to find five articles that fit their criteria. All of the five articles were case studies with one faculty member as a participant, and all used observation and interview data to analyze the instructor's teaching practices. Only one of the articles mentioned was an analysis of a proof-based course (Weber, 2004).

Weber's (2004) study analyzed a professor's use of the 'definition-theorem-proof' model. Although this course was taught in the traditional lecture style, the professor didn't seem to present proofs in a linear style. He made an effort to reveal the reasoning behind the proof construction so that students could learn to construct original proofs themselves. Another recent study examines the pedagogical choices of a faculty member teaching a traditional, proof-based course. Fukawa-Connelly (2010) observed a mathematics faculty member over the course of a semester in an abstract algebra course. This study gives an existence proof that university mathematics professors do not always use a "pure telling" method of proof presentation. He analyzed classroom dialogue through the lens of pedagogical content tools, looking for instances in which the faculty member 'modeled mathematical behaviors.' One of the mathematical behaviors he identified was *proof writing,* which he described as a collection of proof creation heuristics. This study will explore the mathematical behavior of *proof writing* further to identify some strategies that mathematicians use when presenting proofs in class.

There have been several studies that investigate mathematics faculty members' pedagogy in regard to proof presentation (Weber, 2010; Alcock, 2010; Yopp, 2011; Hemmi, 2010). These studies have used interviews of faculty members to investigate why and how they teach proof, and have described how these mathematicians talk about their intentions and pedagogical perspectives.

Two of the studies investigated the reasons that the participants presented proofs to their students. Yopp (2011) interviewed 14 mathematicians and statisticians, and all of the participants mentioned that they present proofs to develop their students' ability to write proofs on their own. No other reason was mentioned so frequently by the subjects. Of the nine mathematicians interviewed by Weber (2010), six of them mentioned that they would present a proof when it illustrates a new proving technique. This may be because their goal is also to equip students to write proofs on their own. Other reasons for presenting proofs included cultural reasons, to expose students to proof, or to illustrate the truth of a theorem.

Alcock (2010) identified four modes of thinking that the faculty members aim to teach their students by presenting proofs. The mathematicians considered these ways of thinking natural for them, but not for students. *Instantiation* is used to "understand a mathematical statement by thinking about its referent objects," (p. 69) and includes thinking about examples or images. Mathematicians often use this mode of thinking when presented with a new definition. *Structural thinking* is a way of reducing abstraction and making use of the logical structures to drive the construction of the proof. *Creative thinking* includes experimenting with examples with the hope to generalize or attempting to construct a counterexample. The goal of *critical thinking* is to check for the correctness of each line of a proof.

Hemmi (2010) outlined three dimensions of presenting proof that seem to encapsulate the different opinions of faculty members. Induction/deduction refers to whether the presentation generalizes from examples or starts with a generalization. Visibility/invisibility refers to the degree to which the logical structures are made explicit. Intuition/formality refers to the amount of heuristic and visual arguments that are used versus purely formal, rigorous proof methods.

Though these studies help to describe how faculty members talk about their teaching practice, they make no effort to describe what is actually happening in the classroom. Hemmi (2010) acknowledges that "the relationship between talk and reality is complex. The mathematicians' talk about proof and the teaching and learning of proof is considered as shedding light on the practice, but not as an objective description of it" (p. 272). This study will address this gap in the literature by collecting and analyzing both interview and observation data from faculty members who are teaching mathematical proof.

Research Methods

In this two-phase study, we first conducted interviews with mathematics faculty members who were teaching a proof-based course to develop a backdrop and framework. By "proof-based," we mean a course in which students are expected to construct original proofs on a regular basis. Secondly, we conducted video-taped observations of the same faculty members throughout the course of the semester. We then analyzed the proof presentations in the video data using the framework that we developed from the interview data.

We seek neither to criticize nor to commend the presentation methods of these mathematicians. Our goal is merely to describe the practices of mathematics professors as they present proofs in a traditional style in upper division, undergraduate level mathematics courses. In our description, we will highlight some of the tools that are being used to present mathematical proof, and catalog the level of engagement that the instructor seems to expect from the students during proof presentations. Our research questions are the following:

1. What do mathematics faculty members contemplate as they plan lectures that include proof presentations?
2. What pedagogical moves do mathematics faculty members make when presenting proofs in a traditional undergraduate classroom?
3. To what degree and in what ways do faculty members engage students when presenting proofs?

This paper will concentrate on the second and third research questions.

Participants: The four participants in this study were tenured faculty members at a large comprehensive research university in the Midwest. All were experienced teachers and researchers, and all were scheduled to teach an upper-division proof-based course. The four participants were interviewed, and three of the participants agreed to allow video-taping of their lectures. The participant who declined to be video-taped was observed, and field notes were collected, however, the researcher did not feel that enough information was gathered to include that data in the results, because the coding scheme had not yet been developed. All of the instructors who participated in the observations identified their teaching style as traditional lecture. The courses in which the observations took place were Introduction to Abstract Algebra, Geometry, and Number Theory.

Because the data were collected at one university, we must take extra care to protect the identities of the participants in this study. Although the mathematicians' gender, age, ethnicity, and area of research may shed some light on data analysis, we cannot reveal this information. We will therefore use pseudonyms, and will use the pronoun "he" to refer to all of the participants throughout the discussion, regardless of their gender.

Phase 1: The first phase consisted of semi-structured interviews addressing what the instructors do when they present proofs in class, why they make those choices, and what they do to help students understand their presentation of proofs in class. The interview data were transcribed and broken into chunks which were sorted into groups to identify themes. Several themes emerged, but two overarching themes were selected to use for analysis of the observations. These two themes were *expected engagement* and *proof presentation strategies*. Both of these themes also appeared in the literature. In the area of geo-science education, Markley, Miller, Kneeshaw, and Herbert (2009) conducted faculty interviews and classroom observations to determine the level of student engagement. Weber (2010) cataloged several strategies that faculty members claimed to use to help students understand proof. Our presentation strategies overlapped with some of Alcock's (2010) modes of thinking and with Mejia-Ramos, Weber, Fuller, Samkoff, Search, & Rhoads' (2010) dimensions of proof comprehension. The interview data were used to construct levels of expected engagement and indicators for each proof presentation strategy.

Phase 2: Throughout the semester, observations of three of the instructors were conducted and analyzed in detail. Each participant was video-taped 6 to 7 times throughout the semester, approximately once every two weeks. The participants communicated exam dates so that those could be avoided, and also occasionally communicated when a particular important proof would be presented. So, the dates for observation were selected by convenience sampling.

Transcriptions were made of each instance of proof presentation, beginning and ending at natural breaks in the dialogue. The researcher chose instances where the professor wrote a formal proof on the board, provided justification for an argument, or worked problems on the level of students' homework. Because the homework problems required proof or some kind of formal justification, we also counted homework problems that were worked or discussed in class as proof presentations.

Once the transcriptions were complete, the researcher read through each observation again, and took note of the amount of time spent on each proof. The times were analyzed to determine the amount of class time spent on proof, the number of proofs per 50 minute class period, and the average amount of time spent on each proof. The researcher then coded each proof according to the *proof presentation strategies* that appeared and the level of *expected engagement*, which will be discussed further in the next section.

Results

As this is a preliminary report, the results presented in this paper represent the framework that we constructed from the interview data and our initial pass through the observation data. This paper will describe the framework that we constructed, but will not contain a detailed analysis of the interview data. Our initial analysis of the observation data determined the amount of class time each professor spent on proof presentations, and these results are shown in Table 1. Next, the observation data were analyzed using the framework we constructed from the interview data. We identified the *proof presentation strategies* used in the proofs, and the level of *expected engagement*.

We identified four *proof presentation strategies* from comments made by the professors in the interview data. We gave them the names *outline, instantiation, logical structure,* and *context.* As we worked to identify these strategies, we echo Yopp's lament that "it is challenging to define the categories in a way that distinctions between them are clear. This issue seems to be heightened when authors discuss instruction" (Yopp, 2011, p. 3). We make no claim that these strategies are mutually exclusive or exhaustive, they merely represent some of the strategies mentioned by our participants in the interviews. We will describe each of these strategies and discuss the indicators that we used in our coding. Although it would be helpful to include instances of each strategy from the observation data, we will not do so in this paper for lack of space.

Table 1: Analysis of class time spent on proof

	Dr. Grey	Dr.	Dr.

		Nelson	Adams
Percentage of class time spent on proof	70.1%	36.1%	42.5%
Average number of proofs per 50 minute class period	3.7	1.3	2
Average amount of time spent on each proof in minutes	7:58	11:33	11:27

When asked what they do to help students understand a proof presentation, several of the participants described a process of outlining or talking informally about the key ideas of the proof *before* they began to write the proof on the board. For example, Dr. Nelson said that before he begins to present a proof in class, he tries to "*read* the… theorem, and I say, 'What's important here? What do you think of when you read this?' And I ask for ideas, what relevant theorems might be true…" We called this strategy *outline.* It sometimes took a form similar to Leron's (1985) top-down approach in which he begins a proof by giving an overview, but other times it was just an informal discussion about how to begin the proof. When coding, we looked for instances where the faculty member discussed with students about the ideas of the proof before starting the proof, and we also looked for when the professor was very clear about what was just informal talk and what was a formal proof.

The mathematicians also mentioned drawing pictures or using specific examples to help students understand the meanings of terms or statements, or to motivate the proof. Dr. Nelson said that "…if it's a proof of a pattern, then I certainly… emphasize computation. First, you have to compute a lot to try to figure out what the pattern is." Although we first identified *drawing pictures* and *using examples* as separate proof presentation strategies, it seemed to be that the amount of pictures or numerical examples used seemed to rely on the mathematical content. In Alcock's (2010) study, she used the term *instantiation* to mean "understand[ing] a mathematical statement by thinking about its referent objects." This terminology seemed much less content specific. In the observation data, we were looking for proofs in which the professor presented algebraic, numerical, or pictorial examples to explain the mathematical statements or the proof strategy.

Though the mathematicians who participated had differing views about whether the logic of proof writing should be taught along with content or separately, they all mentioned that they spend time talking about *logical structure*. When Dr. Adams begins a proof presentation, he says, "I make the statement on the board, and I point out if it's an if-then statement, or if it's a, both, if-and-only-if statement, and I talk about that, what we have to prove then." When analyzing the data, proofs in which the professor used the *outline* strategy often also emphasized *logical structure*, but not always. Indicators of *logical structure* included pointing out when hypotheses are used, explicitly discussing the structure of the statement to be proved, or summarizing the logic of the proof after the proof was completed.

The final strategy that emerged from the interview data was *context.* We used this to describe instances when the professor placed the ideas of the proof in historical context, pointed out standard arguments as they appeared in presentations, or highlighted the key ideas and how they fit within a larger context of the mathematical content area. For example, when describing how he presented the proof that trisecting an angle is impossible, Dr. Grey said that "…the whole idea that you can prove that it's impossible, no matter what you do… that's kind of a big thing to understand right there." When presenting the proof, he emphasized the standard arguments required to show that a construction is impossible.

Each proof was analyzed and coded with the presentation strategies that occurred in the proof. There were very few of the proofs that used none of the strategies, and some that used all four. Although we have not completed a detailed analysis, it is our impression that professors are more likely to employ multiple strategies when the theorem is more important.

Table 2: Percentages of proofs using each presentation strategy

	Dr. Grey	Dr. Nelson	Dr. Adams
Total Number of Proofs in Observation Data	22	8	12
Outline	54%	63%	42%
Instantiation	27%	50%	67%
Logical Structure	36%	50%	25%
Context	27%	38%	17%

By *expected engagement*, we mean the extent to which the professor seemed to expect students to contribute to the proof presentation. We used each individual proof as the unit of analysis, and assigned a code of 1-5 representing different levels of expected engagement. Proofs coded 1 or 2 represented instances where the instructor did not seem to expect the students to be actively contributing to the proof presentation. The code 3 was assigned to proofs in which the instructor seemed to expect students to contribute factual information, and 4 or 5 was assigned to proofs in which the professor expected students to contribute both factual information and key ideas for the proof presentation. To determine the level of expected engagement, we looked for both the types of questions that the instructor posed as well as the amount of time that he waited for a response. For example, if the instructor posed a question and immediately answered his own question, we assumed that he did not expect the students to respond. The coding scheme for expected engagement is summarized in Table 3.

Of the 42 proofs that were video-taped and transcribed, 16.6% of them were coded 1 or 2, 50% of them were coded with a 3, and 33.3% were coded with a 4 or 5. So, this means that in well over half of the proof presentations, the faculty members expected the students to actively contribute to the presentation, whether by providing some factual information or actually helping to contribute key ideas for the proof. The data for each professor is shown in Table 4. Since this was a study of faculty members and not students, there was no data collected about the actual number of students who participated in class, or the other ways in which students were engaging. We were merely documenting the apparent expectation of the professor in regard to student input in the proof presentations.

Table 3: Coding scheme for expected engagement

	Description
1	Professor does not request active contribution during the proof presentation, and does not appear to engage students.
2	Professor does not request active contribution during the proof presentation, but seems to adapt the presentation based on monitoring, non-verbal communication with students, or feedback from closed-ended questions.
3	Professor expects students to contribute some factual information during the proof presentation.
4	Professor expects students to contribute some factual information and some key ideas of the proof during proof presentation
5	Professor expects students to generate the majority of the ideas for the proof during the proof presentation.

We will now give examples of different instances of proof presentations from the observation data and how they were coded for *expected engagement*. Because the video camera was directed at the instructor, any comments from the students are simply labeled 'Student.' It is not necessarily the case that the same student spoke each time. Also, single quotes are used to identify times when the instructor was writing on the chalk board while talking aloud.

The expected engagement code 1 or 2 represents proof presentations where the faculty member did not appear to expect the students to actively contribute to the proof presentation. This does not imply that there was no interaction at all, or that the students were not engaged in other ways such as note-taking or non-verbal communication. In most of these instances, the instructor used monitoring to adapt his presentation. For example, he would ask if the students understand, watch their reactions, and modify the presentation if he deemed necessary. One such example is taken from Dr. Adams' class, where he presented a proof that the composition of homomorphisms is a homomorphism.

Dr. A: What does that mean? 'G, H, um, P groups. $\theta : G \to H$,' and uh, let's see… theta... let's say psi, '$\psi : H \to P$, homomorphisms, then $\psi \circ \theta$ is a homomorphism.' So, we know that composition is an operation on mappings. So, when I say theta and psi are homomorphisms, first of all, they are mappings of the underlying point-sets. So these are mappings. Hence, their composition makes sense. So, their composition is a mapping. So, the question here is… we know this is a mapping *(circles $\psi \circ \theta$).* Is it a homomorphism?

Dr. A: So, 'Proof:' What do you need to do to prove this? Um, well, we have to show that multiplication in G, if you take multiplication and take it under the composition, it is the multiplication of the factors in P, uh, of the images in P. So, we 'Let $g_1, g_2 \in G$,' and we 'Consider $\psi \circ \theta(g_1 g_2)$ ' Ok? What is that equal to? Well, first off, it is equal to ' ' that is… that's what the definition is of a composition. Ok? Theta, though, is a homomorphism. So this is '$\psi(\theta(g_1)\theta(g_2))$' Now, all these guys have big symbols here… lots of symbols. This is a single element in H, *(circles $\theta(g_1)$ with his finger)* this is a single element in H *(circles with his finger)*. Now, would it be better if I put those stars and diamonds and dots in there for everybody, or are you ok with this? Ok? And so we have this. But this now, and we totally forget G for a minute, this is an h_1 *(circles $\theta(g_1)$ with his finger)*, this is an h_2 *(circles with his finger)*, these are just elements in H. The image of the product is the product of the image, because psi is a homomorphism.

Dr. A: So, this is, this uh, theta a homomorphism, and now we do this as '$(\psi\theta(g_1))(\psi\theta(g_2))$' and this is because psi is a homomorphism. But now, again, we go back to the definition, this is '$((\psi\circ\theta)(g_1))((\psi\circ\theta)(g_2))$' and so, if you take this guy on the product, you end up with here, the product of the images. So, here, the image of the product is the product of the images. So, it is *(writes '$\psi\circ\theta$ is a homomorphism')*. Ok?

So, we can see that Dr. Adams did ask some questions, however, he did not wait for a reply from the students. The questions posed were merely part of his presentation style. He may have been modeling the questions that he would ask himself as he wrote the proof, but there did not appear to be an expectation that the students would actively contribute to the proof presentation.

Table 4: Percentages of proofs coded for expected engagement

	Dr. G	Dr. N	Dr. A
Number of Proofs in Observation Data	22	8	12
Coded 1 or 2	4.5%	0%	50%
Coded 3	54.5%	50%	41.6%
Coded 4 or 5	40.9%	50%	8.3%

Proofs coded 3 represent an instance where the professor did expect the students to actively engage in the presentation, but only to contribute factual information. There did not appear to be an expectation that the students would create any of the big ideas for the structure of the proof. An example of this level of expected engagement comes from Dr. Grey's presentation about the summit angles of a half-rectangle.

Dr. G: A half rectangle is a convex quadrilateral with two right angles at the base. And, what I want to prove is that in a half-rectangle, I want to look at these two angles… this is going to look like a test problem before we're through…

If I look at those two angles, I claim that the greater side is across from the greater angle. So, 'In a half-rectangle, the greater side is across from the greater summit angle.' Those are the summit angles. So, what I want to do, what's the theorem? The theorem is angle 1, well, let's do it this way. 'Suppose BC is bigger than AD,' and I want to prove that angle 1 is, uh, bigger than angle 2. Suppose this is true. Ok, Well, let's see, let's 'choose a point *E* so that *B*E*C* and *BE* congruent to *AD*.' So, pick a point so that those two things are congruent. What kind of an animal is this? *DABE*? It has a name.

Student: Saccheri rectangle?

Dr. G: It's not a Saccheri.. don't say rectangle.

Student: Saccheri Quadrilateral?

Dr. G: Saccheri quadrilateral. Ok? 'So, *ABED* is a Saccheri quadrilateral.' What do I know about summit angles of Saccheri quadrilaterals?

Student: They're congruent?

Dr. G: Yeah. We proved that on a test, I think, or on a previous test, or a homework. 'Angle 3 congruent to angle 4.' Oh, now it's gonna look like the test we just did, because angle 1, degree measure, is bigger than angle 3, which is the same in degree measure as angle 4, and this angle is… how does that compare with angle 2? How does angle 4 compare with angle 2?

Student: It's exterior.

Dr. G: It's exterior, isn't it? It's exterior to this triangle. *(re-draws part of the picture to emphasize that angle 4 is exterior)* Angle 4 is exterior to the triangle, so it's, um, bigger than angle 2. Ok.

In this proof, we can see that Dr. Grey expected the students to respond to most of his questions. When he is outlining the structure of the proof, he seems like he is going to ask the students how to begin the proof, but he doesn't wait for their response. He then lays out the beginning of the proof, and begins to ask the students factual questions along the way. We believe that he expects the students to respond to these questions because he asks the same question multiple times, and waits for a student response. When Dr. Grey hears the desired response, he takes that information and immediately proceeds to the next step in the proof. The students do dialogue with Dr. Grey, and they do contribute to the proof, however, Dr. Grey is clearly in control of the construction of the proof.

Since the expected engagement level of 4 or 5 may differ slightly from the way we conceptualize lecture style proof presentations, we included three examples, one from each faculty member. This excerpt from Dr. Nelson's class was coded 4, because the students give the key ideas for how to construct a proof of uniqueness.

Dr. N: Um, all right, well, unfortunately after all that success, you still have uniqueness to prove. And, um, you'd like to think that maybe uniqueness was part of what you were doing, but it's really hard to justify that. So, the best way to prove uniqueness is to… I mean, what's the standard way that you prove that there's only one formula? There's sort of a standard strategy that you do. When you've done this already… we did this for the division algorithm. You remember what I said? At the end there's a uniqueness part of the division algorithm, and… how did we prove that uniqueness was actually true? Yeah, so let me write up here, 'proof of uniqueness' And there's always sort of a standard way. So, you've proved that there is a formula, and you want to show that that formula is unique, so what's the strategy for doing it?

Student: Assume there's two, and show it isn't possible.

Dr. N: Exactly! All right, that's all I want you to say is just assume that there's two. Assume there's two and then show that they are actually the same. So, 'Suppose there are two such formulas for the same n.'

Student: So are we just proving contradiction?

Dr. N: Yeah, this would be an example of proof by contradiction. What's the opposite of uniqueness? The opposite of uniqueness is that there's more than one. So, I'm going to assume that there are… this is kind of a specialized version of proof by contradiction. It always works for uniqueness. You always assume there are two, and then you try to prove that they are actually the same. So, suppose there are two such formulas, um, so the first one we got was that over there, let me just summarize it, ' ' and for the second one, we have to use a different letter for the coefficients, b is the same, but I'm going to change the a to a c, and the k to an l, because it's possible that if I did it in some other crazy way, maybe I got more, uh, more digits or something. '$n = c_l b^l + ... + c_1 b + c_0$' And now the strategy is to prove that something is wrong with this. Suppose there are two different ones. Suppose they are actually different. We have two different formulas. Now, how would I know that these are really different? What specifically would.. what would happen that would tell me that these are really different? Now, you sort of have to hone in on what's the difference between the two formulas. You know, what's the specific difference. Now, if they are really different, something has to happen.

Student: *(mumbles)*

Dr. N: Now, you're saying *a*, but there are a lot of *a*'s, you have to be specific.

Student: a_j is not equal to c_i, or c_j.

Dr. N: No, that's not correct, or, what did you say at the end?

Student: c_j

Dr. N: c_j, ok, right, ok? So, it doesn't matter that this digit is not equal to that one, it's the corresponding ones. So, what Nathan is saying here is that '$a_j \neq c_j$ for some j.' Not for all of them, but for one of them, they have to be different. Ok, now this is really hard to motivate until you've seen this kind of proof, but, one of them, I know that in one of them they are different, but I want to pick, I want to sort of be real specific about j. I want to say, 'take j to be' Now, j could be zero, it could be 1, it could be anything in those ranges there, but I want to work specifically with j having a real specific property here. You know, until you get into this, it's really hard to motivate. Do you know what I want j to be? I want the two digits to be different, but I want to know something more about j. *(mumbles)* You have to learn this…

Student: The smallest one?

Dr. N: The smallest one, there we go. You got it. You know, when you're trying to hone in, you always want to go to an extreme case…

We can see that Dr. Nelson continues to ask the students questions about how to structure the proof, and asks the same question in different ways until the students give an answer. Although the students are contributing significant ideas to the proof, this presentation is still teacher-centered. When the students give the expected answer, Dr. Nelson spends a little bit of time explaining why the answer that the student gave is correct, and then carries on with the proof.

One common approach in Dr. Grey's class was to have the students identify the contradiction at the end of an indirect proof. The students didn't structure the proof, but they did contribute a key idea to the proof by identifying the contradiction. Here is an example of this type of engagement.

Dr. G: So, I was going to say, we did this proof before, but I was just going to remind you how it goes. How do I show that if Euclid 5 is true, then the converse to alternate interior angles is true? This is what I want to prove *(underlines 'converse to AIA' writes 'prove this' underneath)* But, since I'm assuming Euclid 5 is true, I get to use this. *(underlines 'Euclid 5' and writes 'Use this in my proof' underneath.)* So, let's see, I want to show the converse to alternate interior angles is true, so I say, 'suppose that k is parallel to l,' and I need to 'prove that angle 1 is congruent to angle 2.' So how do I do that? Well, suppose it's not. 'Suppose angle 1 is not congruent to angle 2.' Well, then what are we going to do, is use my protractor axiom to reproduce angle 1 right here. I'm running out of letters. n. So, let's choose n through Q so that angle 3 is congruent to angle 1.

Student: So, you're saying that angle 1 is bigger than angle 2?

Dr. G: This picture makes it look greater, It would, it's gonna be the… it's gonna be the same argument if it were smaller. So, I'm just assuming they're different. Ok, I claim that we're going to get a contradiction here all ready. I claim I'm done. Where is my contradiction? Well, what do I… let me… what do I know about n and k?

Student: They're lines.

Dr. G: They're lines, I do know that. That's correct, but I would hope to know more.

Student: Parallel.

Dr. G: Why?

Student: Alternate interior.

Dr. G: Alternate interior angles? Ok, so 'k is parallel to n by alternate interior angles.' It's not the converse, but the theorem itself.

Student: And then Euclid 5, there's a contradiction because it says there's a unique parallel.

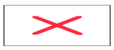
Dr. G: Right. I get to use Euclid 5 in my proof, and now I've just built two lines parallel to k through the point Q. Now, 'both l and n are parallel to k and through Q.' That's a contradiction. So, that didn't happen. Those angles were the same.

Dr. Adams frequently answered questions about homework problems at the beginning of class. On one such occasion, we can see that the professor expected students to be involved in both example generation and a discussion about the necessity of the hypotheses in this particular proof.

Dr. A: Uh, '*a* equivalent to *b* if and only if for some . A)

Prove that this is an equivalence relation,' and 'B) give a complete set of equivalence class representatives.' So, you had a problem with showing that it's an equivalence relation, right? And, so *a* is equal to *b* to the *10 k* for some *k*, so, um, we, uh, have to have, then, so let's see. So what's the equivalence class, say, of… what are *a* and *b*?

Student: Natural numbers

Dr. A: *(writes*  *at the top)* So, let's see. What's equivalent to 1? Yeah, we want a complete set of equivalence classes.

Student: 1.

Dr. A: Well, 1 is. That's a good start. What else?

Student: 10.

Dr. A: Right. So, 1, 10, uh, and then we go 100, … and then we go 10^k. *(writes {1,10,100,…* 10^k *…})* '*k* belonging to *Z*.' So, I should maybe, uh, do it the other direction too, so since we have k negative, um, let's see…

Student: Can *k* be negative, though? I don't think *k* can be negative, because then it makes a… not a natural number.

Student: It can be negative if *b* is greater, like a multiple of 10.

Dr. A: So, did it say some *k* belonging to *Z*?

Student: Yes.

Dr. A: So, *k* has to be greater than, *k*, and that, uh, greater than zero. Ok, so that's…

Student: So that's…

Student: It can be zero.

Student: It doesn't have to be greater than zero. If *a* is a multiple of 10 then *b* can be, like 4, and then, whatever, and then it just decreases… Does anybody else see what I'm saying?

Student: If *k* is not equal to zero, we don't have the reflexive property of 1 being equivalent to itself.

Dr. A: Ok. *(erases k>0)* So, at least, for a natural numbers, things that are equivalent to 1 should be that. Do you agree with that? And now, 2, we get, what? 20, 200, so on and so forth. Right? *(writes {2,20,200,…2* $\cdot 10^k$ *…})*. Now, for general *n*, what would we get?

In this instance, Dr. Adams was posing questions and actively listening to the students' responses. The entire class was exploring the mathematics together, and Dr. Adams was merely facilitating the discussion. This may be because the student asked a question about a homework problem with which Dr. Adams was unfamiliar, and so he himself was working to make sense of the problem. This discussion continues until the class comes to an agreement about the possible values of *k* and how they are used in constructing a proof that this relationship is an equivalence relation, but they do not formally write down the proof because it was a homework problem.

We can see that in all of these instances the mathematician engaged the students, and seemed to expect that the students contribute key ideas to the proof. So, we have begun to examine the strategies that mathematicians mentioned in the interviews, and how they manifest themselves in the classroom.

Limitations of Study

This study analyzed several observations of each faculty member periodically throughout the course of a semester. The mathematicians in this study were teaching different courses in different content areas, so naturally the content could influence presentation strategies. We are not attempting to compare the instructors to each other across different content areas, but are seeking to describe the presentation strategies that each used in their classroom. Although we did make an effort to choose proof presentation strategies that were content independent, it may not be possible to completely divorce the content from the presentation style.

The observation data reflects the manner in which the material was presented during the semester in which the observations were made, but may not reflect the past or future behavior of the faculty member involved. The proof presentation strategies used and expected engagement level could depend on several factors, such as the ability levels and experience of the students in the classroom, or the classroom dynamic created by each particular group of students. The interactions between the instructor and his students may also reflect the instructor's current mathematical content knowledge and pedagogical content knowledge, which both change over time.

The subjects of this study were faculty members, and therefore data was not explicitly collected from the students. We have no data on the number of students who interacted with the faculty member or ways in which they engaged.

The results presented here were coded by one researcher independently. It would increase the validity of this study if more than one researcher participated in the coding. Also, the coding was based only on the instances of proof presentation, so it is possible that other strategies were used by the instructor when not explicitly presenting proofs.

Conclusions

As we work to describe how faculty members present proofs in class, our goal is to contribute to the a foundation of knowledge about how proofs are presented in the undergraduate mathematics classroom. Although much more work needs to be done, we can draw a few conclusions from the preliminary analysis of our data.

First of all, we can conclude that faculty members in a traditional lecture classroom expect engagement from their student, and that the level of engagement during proof presentations varies. Even in the early stages of analysis, we can already see that these instructors do expect students to contribute the proof presentation. All three professors presented proofs with the expectation that the students would provide factual information during the proof presentation in approximately half of the presentations, and they also expected students to contribute key ideas the proofs in many of the presentations that were in our video data.

Another conclusion that we can draw from this data is that the faculty members are not only talking about strategies for proof presentations, but are actually using the strategies that they talk about. The proof presentation strategies that we identified from the interview data were not entirely unique to this study, but bore similarities to the strategies mentioned in the literature (Weber, 2010; Alcock, 2010; Mejia-Ramos, et. al., 2010). We have documented that these instructors not only think about the pedagogy of proof presentation, but that they use specific strategies with the goal of enhancing student learning.

We have seen that the lecture style of proof presentation is richer and more diverse than is currently documented, and can include significant levels of student engagement. Future directions for this research will include follow-up interviews with the faculty members. This data will strengthen the validity of our study and may give us ideas for different ways to look at our video data. It may also be interesting to collect data from the students who are attending to and participating in the proof presentations. We will continue to analyze the data from different perspectives to draw out other facets of proof presentations in the classroom.

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