Students' Reinvention of Formal Definitions of Series and Pointwise Convergence

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The purpose of this research was to gain insights into how calculus students might come to understand the formal definitions of sequence, series, and pointwise convergence. In this paper we discuss how one pair of students constructed a formal ε-N definition of series convergence following their prior reinvention of the formal definition of convergence for sequences. Their prior reinvention experience with sequences supported them to construct a series convergence definition and unpack its meaning. We then detail how their reinvention of a formal definition of series convergence aided them in the reinvention of pointwise convergence in the context of Taylor series. Focusing on particular x-values and describing the details of series convergence on vertical number lines helped students to transition to a definition of pointwise convergence. We claim that the instructional guidance provided to the students during the teaching experiment successfully supported them in meaningful reinvention of these definitions.

Keywords: Reinvention of Definitions, Series Convergence, Pointwise Convergence, Taylor Series

Introduction and Research Questions

How students come to reason coherently about the formal definition of series and pointwise convergence is a topic that has not be investigated in great detail. Research into how students develop an understanding of formal limit definitions has been largely restricted to either the limit of a function (Cottrill et al., 1996; Swinyard, in press) or the limit of a sequence (Cory & Garofalo, 2011; Oehrtman, Swinyard, Martin, Roh, & Hart-Weber, 2011; Roh, 2010). The general consensus among the few studies in this area is that calculus students have great difficulty reasoning coherently about formal definitions of limit (Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). The majority of existing research literature on students' understanding of sequences and series concentrates on informal notions of convergence (Przenioslo, 2004) or the influence of visual reasoning or beliefs (Alcock & Simpson, 2004, 2005). Previous literature on pointwise convergence has been in the context of power series addressing student understanding of various convergence tests (Kung & Speer, 2010), the categorization of various conceptual images of convergence (Martin, 2009), the influence of visual images on student learning (Kidron & Zehavi, 2002), and the effects of metaphorical reasoning (Martin & Oehrtman, 2010). Two student volunteers from a second-semester calculus course participated in a teaching experiment with the goal that they reinvent the formal definitions of sequence, series, and pointwise convergence. For this paper we posed:

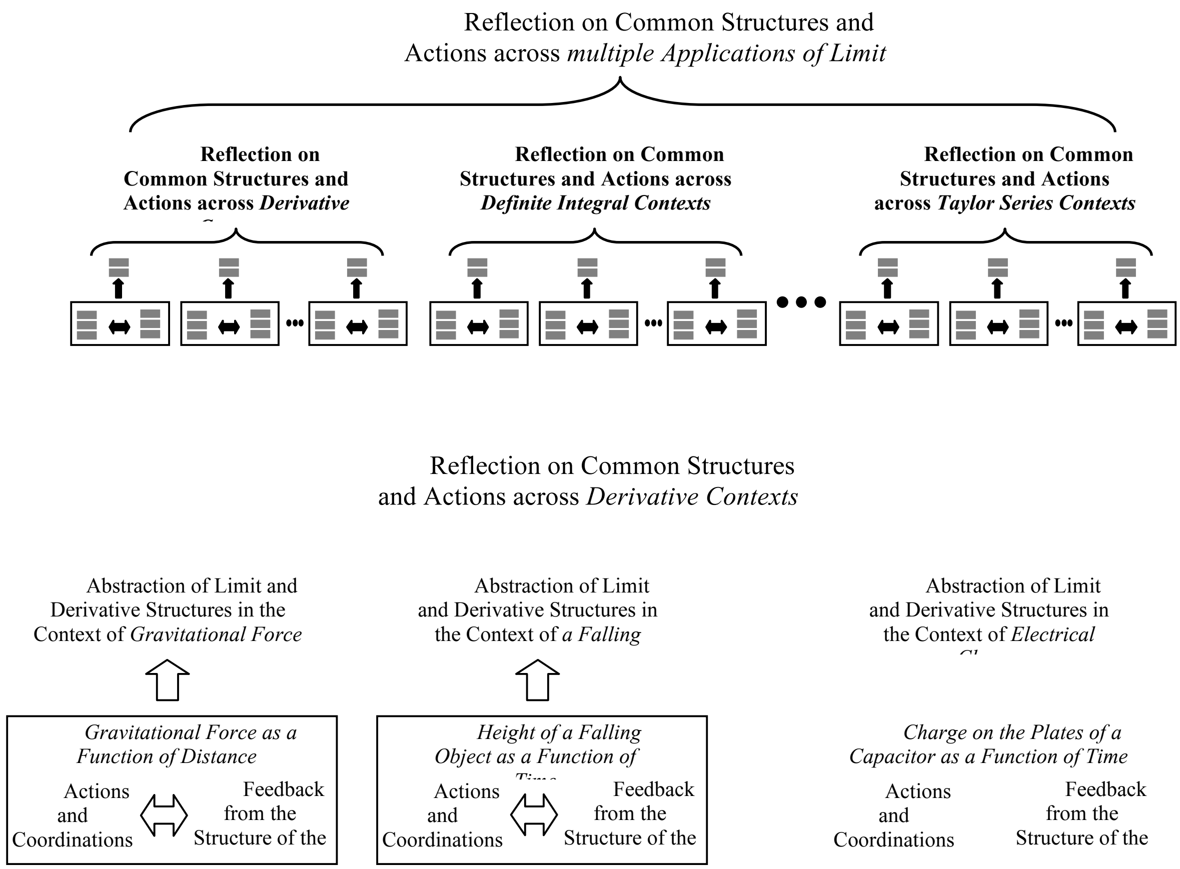
1. What are the challenges that students encountered during guided reinvention of the definitions for series and pointwise convergence? How did students resolve these difficulties?
2. What aspects of the students' definition of sequence convergence supported their reinvention of series convergence? What aspects of the students' definition of series convergence supported their reinvention of the definition of pointwise convergence?

Theoretical Perspective

To investigate our research questions, we adopted a *developmental research* design, described by Gravemeijer (1998) "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (p.279). Task design was supported by the *guided reinvention* heuristic, rooted in the theory of Realistic Mathematics Education (Freudenthal, 1973). Guided reinvention is described by Gravemeijer, Cobb, Bowers, and Whitenack, (2000) as "a process by which students formalize their informal understandings and intuitions" (p.237). The design of the instructional activities was inspired by the *proofs and refutations* design heuristic adapted by Larsen and Zandieh (2007) based on Lakatos' (1976) framework for historical mathematical discovery.

Oehrtman et al. (2011) further characterize reinvention in the context of students creating a formal definition as beginning with exploration of a rich set of examples and non-examples, then moving to an iterative refinement process that involves definition creation, checking examples, conflict acknowledgement, and discussion. During the multiple iterations of this process, students engage challenges that arise from their articulations of their definitions trying to formally capture their understandings of a particular concept. These challenges that students face during reinvention are identified by Oehrtman et al. (2011) as opportunities for learning. They identify two basic types of opportunities for learning: *problems* directly identified by students and *problematic issues* unbeknownst to the students but would be identified as problems by the mathematics community. Problems most commonly arise due to conflicts between the students' concept image and their currently stated definition. Furthermore, concerning the currently stated definition, experts might identify other problematic issues that have not been recognized by the students. Problematic issues may become identified problems by the students or may be indirectly resolved by the students while addressing another problem. Oehrtman et al. (2011) claim that it is the thoughtful resolution of identified *problems* that are most meaningful for students and therefore, support the formation of integral ideas that can lead to components of their definitions that remain stable throughout the remaining iterative refinement process.

Students were recruited from a second-semester calculus course covering topics such as sequences, series, and Taylor series, in addition to other topics typical to a second-semester calculus class using the textbook by Smith and Minton (2007). Furthermore, this course utilized Oehrman's (2008) approximation and error analysis framework as a coherent instructional approach to developing the concepts in calculus defined in terms of limits. This framework is based on an approximation metaphor identified by Oehrtman (2008, 2009) as including structures involving "estimates," "error," "accuracy," etc. which involve an unknown actual quantity and a known approximation. For each approximation there is an associated error and a bound on the error. The approximation is viewed as being accurate if the error is small. Actual student usage of the metaphor can be idiosyncratic (Martin & Oehrtman, 2010) but with repetitive usage of ideas of approximations, errors, and bounding errors to reinforce common limit structures within and across different limit contexts, students' use of the metaphor can become more systematized with a metaphorical structure that reflects the structure of formal limit definitions (Oehrtman, 2008). Conversely, a more systematic metaphor can encourage the abstraction of a common structure while engaging in multiple activities within a limit context and the results of such abstractions further support abstractions of common structures across different limit contexts that can provide a more coherent understanding of the role of limit throughout all of calculus and beyond (*Figure 9*). As a student's approximation schema becomes well organized and as more abstract understandings of limit emerge over a long period of time,



this instructional approach can lend itself to supporting students in constructing a more formal understanding of the limit concept through reflection upon common structures and actions performed on approximation tasks.

Methods

The authors conducted a six-day teaching experiment with two students, Megan and Belinda (pseudonyms), who had not received instruction on formal definitions for *sequence convergence*, *series convergence*, and *pointwise convergence*. The central objective of the teaching experiment was for the students to generate rigorous definitions for these three convergence definitions. The full teaching experiment was comprised of six, 90-120 minute sessions with this pair of students after the students had completed their class activities, including a chapter exam, on sequences, series, and Taylor series. The research reported here focuses on the evolution of the two students' definitions of series and pointwise convergence over the course of the 3rd through 5th

Figure 9. Layers of Abstraction

* From Layers of Abstraction: Theory and design for the instruction of limit concepts by Oehrtman, M.

sessions of the teaching experiment following the students' reinvention of a formal definition of sequence convergence.

During the first three days of the teaching experiment, the students engaged in the reinvention iterative refinement process in trying to produce a sequence convergence definition consistent with their concept image of sequence convergence (Oehrtman et al., 2011). Prior to the beginning of the teaching experiment, students from the calculus class engaged in a classroom activity to create an extensive list of graphical images depicting what they viewed as qualitatively different examples of sequences converging to 5 and sequences not converging to 5. Following the production of these graphs, in the teaching experiment Megan and Belinda (pseudonyms) were prompted by the facilitators, Craig and Jason, to construct a convergence definition for sequences by completing the statement, "A sequences converges to U as $n \to \infty$ provided…" Using their list of examples to test their definitions, over the next three sessions the students produced more than 23 definitions for sequence convergence before producing a definition for sequence convergence that they felt correctly captured the meaning of sequence convergence: "A sequence converges to U when $\forall \varepsilon > 0$ there exists some $N \geq 0$ $\forall n > N$ $|U - a_n| \leq \varepsilon$."

In leading up to producing their final definition Megan and Belinda either directly or indirectly engaged many challenges that provided them opportunity for learning through the thoughtful resolution of identified problems. As described in Oehrtman et al. (2011), some of these problems included:

1. How is our definition going to incorporate convergent sequences with "bad [random] early behavior?"
2. What is "infinitely close?"
3. A fixed error bound allows non-examples to be included as limits?
4. We have multiple uses of our "*n*" notation in our definition.

While wrestling with these problems, students had to mentally juggle ideas connected to indices, terms, errors, bounds on errors, universal and existential quantification, notational issues, graphical and formulaic representations, etc. It is after having thoughtfully resolved these issues in the production of their sequence convergence definition that the students were asked to produce convergence definitions for series and afterward, Taylor series, the two definitions that this report describes.

The teaching experiment activities on series began with Megan and Belinda producing and subsequently unpacking details of convergent series graphically. They were then asked to generate a definition by completing the statement, "A series converges when…" After these students had produced a definition of series convergence that they felt appropriately captured convergence in this context, the facilitators guided the discussion to issues of pointwise convergence. To address pointwise convergence, the students were asked to produce a graph of e^x with several approximating Taylor polynomials so that the graph would later be available for students to refer to when producing their Taylor series convergence definition. Later, the students were prompted to talk about what Taylor series were, and finally instructed to produce a definition for Taylor series convergence taking into account their Taylor series graph for e^x. The majority of each session consisted of students' iterative refinement of a definition and the unpacking of their intended meanings for individual elements within each definition.

As described in Oehrtman et al. (2011), there was five-member research team, with two researchers, Jason and Craig, serving as facilitators during the reinvention process. The team debriefed after each session and made adjustments to the next session's protocols as they made hypotheses about the progress of the reinvention. After the sessions were completed, the team transcribed video tapes and created content logs detailing each session with theoretical notes about student progress. A timeline of problems students were addressing was created so that we could attempt to determine both the origin and the resolution, if any, of a given problem.

Results

In this section we share some of the more illustrative examples of student interactions during the iterative refinement process of producing series and pointwise convergence definitions. Examples are chosen so as to illustrate 1) an overview of the reinvention process, 2) some problems students faced, whether directly identified by the students or not, 3) the process of coming to a solution to these problems, and 3) the influence of their prior reinvention experiences during the formation of these new definitions.

Series

In the 3rd session of the teaching experiment, the students initially drew a graph of an alternating series (top graph in *Figure **10***), but as they attempted to recall formulas for convergent series their production of additional series graphs stalled. When the facilitators prompted the students to not focus on finding formulas, the students compared sequence graphs to series graphs and expressed that series graphs "are harder to throw out there." After they stopped focusing on finding a formula they were able to produce a series graph increasing toward 7 (bottom graph in *Figure **10***). During this day, the students' initial and unprompted definition of series convergence to 7 was simply that a series converges when "the a_n's are going to 0 and s_n's are going to 7."

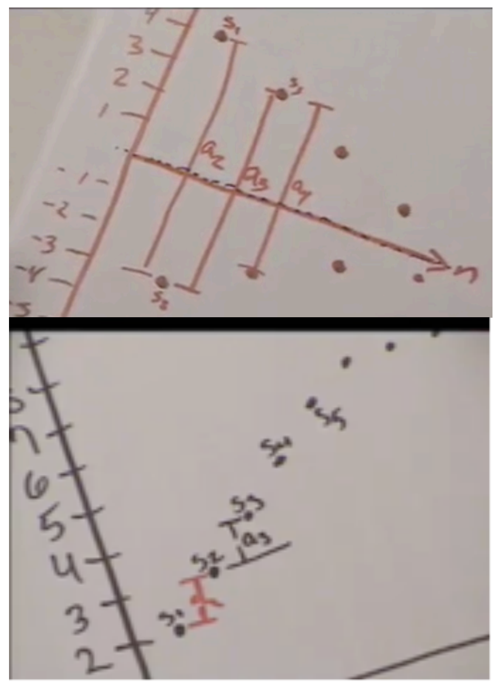


Figure 10. Examples of convergent series produced by the students.

During the 4th session, after being given the prompt to produce a series convergence definition, the students looked at their graphs of series convergence from the previous day and then immediately started making connections to their prior reinvention of sequence convergence.

Megan: Well, basically we could go with what we were talking about before. Only change it to, instead of the approximations, it's the partial sums, more or less. That would be a place to start.

Belinda: Yeah, 'cause I can see it [*looking at their graphs in Figure **10***]. It would be very similar to, to the sequences because you could still set the error bound within a certain, whatever range you want, any error bound,-

Megan: Uh-huh.

Belinda: -and then determine the point N where all the partial sums are within the error bound.

Megan: Uh-huh. Because graphically it looks more or less the same but instead of individual points, they are actually summations of the previous [terms], so instead of approximations, they're partial sums.
Belinda: Well, they're still technically approximations –
Megan: Yeah, yeah.
Belinda: -but they're, just instead of terms –
Megan: -but they're determined by summing the previous.

Belinda saw this activity as "very similar to" defining sequence convergence because she could still choose "any error bound" she wanted and that error bound would "determine" the N after which "all the partial sums are within that error bound." In these shorts statements, Belinda made references to N, ε (consistently attributed to "error bound"), universal quantification of ε, and the relationship of N to partial sums and ε's. Megan agreed, adding that "graphically" series "are more or less the same" to sequences. Furthermore, it should be noted that these students were evoking approximation language as they were reinterpreting the roles of terms, ε, and N in the context of series convergence.

Within 2 minutes they had written the definition, "A series converges to U when for all $\varepsilon > 0$ there exists a value N where all partial sums after N are within ε found by $|U - S_n| \leq \varepsilon$." For an hour, Megan and Belinda interpreted their definition graphically and formulaically using language, including approximation language, and notation from their prior reinvention experience. For instance, they reinterpreted elements, such as N, error bounds, and quantifiers from their prior definition of sequence convergence now as elements in a definition for series convergence. For example, while explaining their series definition they produced another convergent increasing series graph (*Figure **11***), and using this graph, they explained the effect of smaller error bounds on N:

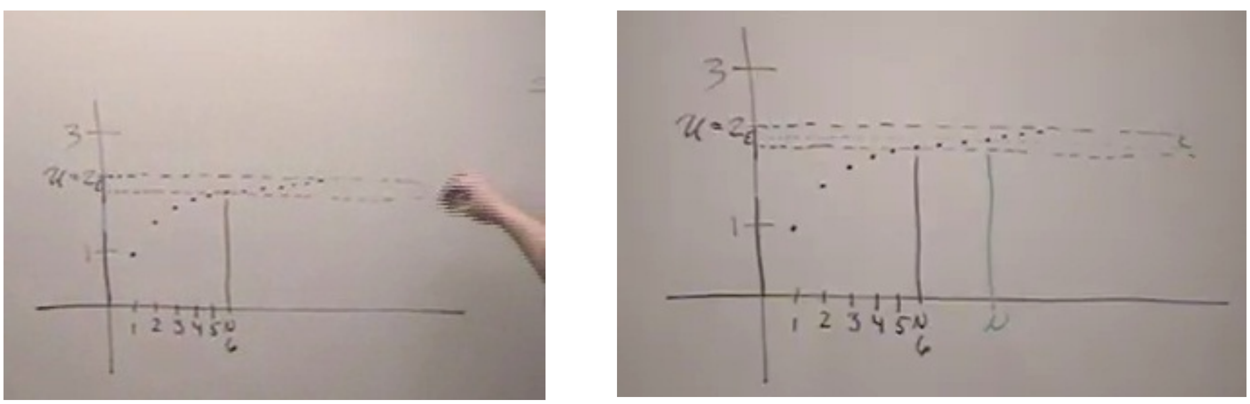


Figure 11. The effect of smaller error bounds on N

Belinda: So then I see how this one works because once we pick any sigma [*pointing to epsilon range depicted by the two horizontal dashed lines in the right graph in Figure **11***] we could move the sigma [epsilon] up or down. It wouldn't matter. For all of them, that exists we would have an N value [*see right graph in Figure **11***] where all the partial sums are within epsilon.

Megan: If we choose an epsilon [*drawing the a dashed green line appearing very faintly between the two horizontal dashed black lines found on the left graph in Figure **11***] here instead… that would make that our new *N* [*drawing the vertical line and the N in green in left graph in Figure **11***]. But after that *N*, these [*pointing to the partial sums to the right of her new N*] are all, still be within that new smaller [epsilon].

Furthermore, they recognized the need to replace their a_n notation for terms with the partial sum notation s_n. However, the students did not *just* change a_n to s_n, they considered each dot in the series graph as representative of partial sums, "You're adding a_1 to a_2 to a_3 to get each one of these dots on the graph of a series (as they illustrated in *Figure 12*)."

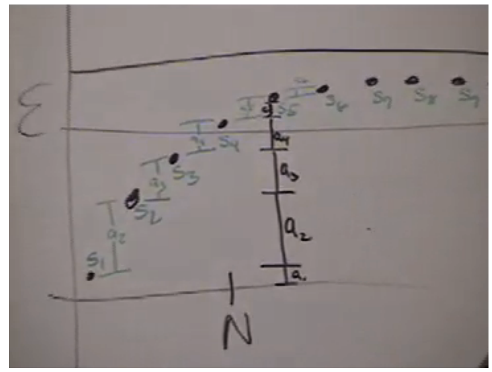


Figure 12. Dots are more than just dots

It should also be noted that they were directed to discuss their definition in the context of the divergent harmonic series. They initially recalled how their sequence convergence definition properly excluded sequences not converging to a given limit candidate when the sequence was in fact converging to something else. In light of this, they went on to explain how their series definition would exclude the harmonic series as being convergent to any particular given limit candidate because an *N* would not even exist for a given error bound.

After a few revisions, they constructed a definition for series convergence as follows: "A series converges to *U* when $\forall \varepsilon > 0$, there exists some *N* s.t. $\forall n \geq N \;\; \left|U - S_n\right| \leq \varepsilon$."

Taylor Series

In initially discussing Taylor series during the 5th session, the students employed informal reasoning as they described various graphical attributes of Taylor polynomials approaching e^x. Following the first prompt, "What is a Taylor series?" students began talking about Taylor polynomials and their various approximation properties; such as how increasing the value for the index yields better approximations and the advantage of re-centering to produce better approximations with smaller values of the index for values of the independent variable away from the center of the series. Some of these discussions appear consistent with formal theory but their conceptions lying underneath their language was not consistent with formal theory.

The sameness of Taylor polynomials and the approximated function. During the later part of the discussion mentioned above, the following exchange occurred:

Megan: I think further out in the series it was the same as the function in question,
Belinda: Yeah, it actually became the function.
Megan: -for more of the graph. […] So like the first one was just tangent or something.
Belinda: Umm-hmm.
Megan: Or the second one was tangent and then other ones kind of started tracing along the graph-
Belilnda: Yeah.
Megan: -for further and further out. […]
Belinda: It was just becoming a way to actually create that function-

Megan: Umm-hmm.
Belinda: -the more n you went out. And depending upon how far you needed to be for that function, you could have less or more n to approximate it, to actually be that function.

Note how for Megan "the series" was the "same as" the approximated function and how the series "starts tracing along" "further and further out" as she moved from one Taylor polynomial to the next Taylor polynomial. For Belinda the series "actually became" and "actually created" the approximated function. Following these comments, Belinda went on to graph sin x and approximating Taylor polynomials. With this graph in front of them, Megan and Belinda continued to reiterate the sameness of the Taylor polynomials and the approximated function at more than just the center of the series. Belinda referred to Taylor polynomials as "actually looking like the sine graph." When asked what she meant, Megan injected that they "have the same values." Belinda agreed and added that the polynomials "fit the sine curve more and more" to the extent that one "couldn't distinguish the difference between the sine curve and Taylor [polynomials] at a certain point." Belinda illustrated what she meant by "certain point" by using 2π and concluded that after "some term" in the Taylor series, "the sine graph between 0 and 2π would be indistinguishable from the Taylor [polynomial]."

After the facilitators then moved the students to looking at their prior graph of e^x and illustrating errors at x=1 using their graphed Taylor polynomials (see *Figure 13*), Craig asked the students what would happen if "we kept taking approximations?" Belinda quickly concluded that "the approximation would equal e" and Megan agreed. Belinda then later clarified and said that "if you take out enough partial sums, then it, then the approximation will equal e^x at x=1." Craig then moved the students to talking about errors.

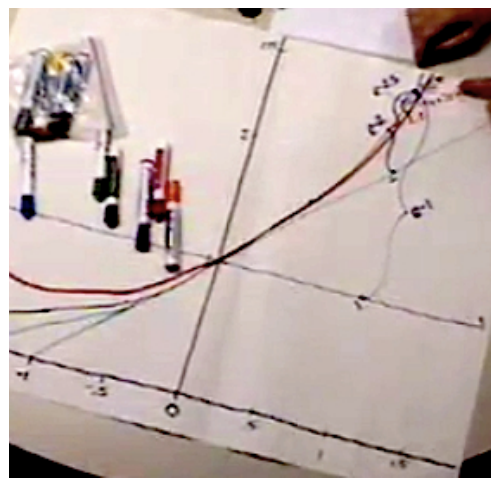


Figure 13. Errors Illustrated at x=1 on Taylor Series Graph for e^x

Craig: So if the approximation equaled e, what would the error be then?
Megan: Zero.
Belinda: It'd be zero.
Craig: Okay, and you're saying eventually your error will be literally zero. When do you think that would happen?
Megan: Based on where we're at right now, maybe another two approximations? If that?
Belinda: Yeah, it's possible. Two or three, I think. Maybe not even two or three. It could just be one more.

Even though their focus had moved to a particular x-value, they continued to employ informal reasoning that entailed Taylor polynomials and generating functions as being exactly the same after a certain number of terms had been added. Once they had used a Taylor series equation for e^x to find an explicit series for e they started to realize that a finite number of terms merely approximated e^x because the remaining terms not used in the approximation had value. Belinda was the first to come to this recognition after Craig highlighted the tail of the series shown in *Figure 14*. Belinda stated that "[the tail] is still technically not really zero" even though "it's very, very small." She then went on to say:

> Like up to here would be a good approximation [*underlining the partial sum for e*] but everything else, this total sum [*underlining the entire sum and putting an arrow on the right*] of everything would converge to *e* when *x* is 1. So this [*pointing to the partial sum for e*] just approximates *e*, but adding everything [*hand spread wide over the series as depicted in Figure 14*] to infinity converges to *e*.

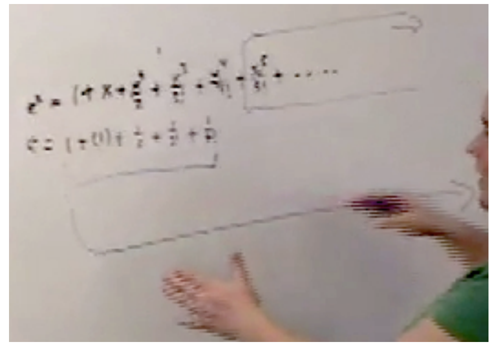


Figure 14. Tail of the Taylor series highlighted by Belinda as "not technically zero"

Belinda did not again make reference to Taylor polynomials being identical to the approximated function away from the center. When Megan later made a reference to a Taylor polynomial being "exactly" e^x at x=0.5, Belinda then proceeded to explain what it meant to "basically converge" at different values for x. Her explanations included references to polynomials getting "closer and closer" to e^x evaluated at corresponding x-values, errors getting "smaller" as more terms are added, and equality only achieved with the "total sum." Notions of polynomials now being exactly the same as the approximated function were missing from Belinda's explanations to Megan. Then, like Belinda, Megan's language started to embody the same ideas and Megan did not again make reference to Taylor polynomials as being identical to the approximated function away from the center.

Initial comparisons between series and Taylor series. When Belinda initially brought up "convergence," one of the facilitators asked, "What do you mean by converges to *e*?" Megan then began recalling those "definitions we wrote" for "sequence and series convergence." The students then made comparisons between series and Taylor series that mainly focused on the influence of the independent variable found in Taylor series but absent in series. For one, "series converge to a number but Taylor series converge to a function." Belinda even observed that Taylor series is "a bit more generalized" than series because "it doesn't have just one number that the Taylor series is converging to." Plus, according to Belinda, Taylor series is "a bit more flexible [than series] because you can either talk about the entire function or you can talk about specific points within the function." It was during these initial comparisons that a first attempt at a Taylor series convergence definition arose from Belinda: "If you add every partial sum going to infinity, then the Taylor series converges to whatever function it's approximating."

Confusion between the independent variable *x* and the index *n*. Following all the conversations depicted above, the students were given the prompt to complete the definition "$1+x+x^2/2+x^3/3!+x^4/4!+\cdots$ converges to e^x provided…" Immediately Belinda said, "Well, we could use the similar idea from sequence and series definitions." And then "given that this is a Taylor *series*, that [*pointing to their series convergence definition*] might be the better one to start with" (emphasis in original). So Megan began to write a Taylor series convergence definition but stopped after writing, "The Taylor series converges to *f*(*x*)." Meanwhile Belinda had been reading their prior series convergence definition and expressed concern over the role that *N* would play in a Taylor series convergence definition.

Belinda: I'm also thinking like we can't really use the *N*, like the cap *N* idea.
Megan: Yeah.
Belinda: Because there isn't really, I don't really see a cap *N* happening. You know what I mean?
Megan: Yeah.
Craig: What do you mean you don't see cap *N* happening?
Belinda: Well with a regular series it was easy to say at this cap, you know, if you set an error bound at-
Megan: There exists.
Belinda: -some cap *N*, […] All of the terms are going to be within that error bound. But this, the graphs look very different, so if we tried to use that, people aren't going to know what the heck is going on.

After these comments Belinda continued to reiterate how the graphs of series and Taylor series look very different. Not only in the approximations but in what they are converging to, "like a regular series graph isn't going to look like a func-, isn't going to be converging to a specific function." Belinda then turned around to their previously drawn Taylor series graph for e^x (see *Figure **15***):

Belinda: So if I used an *N*, like if I said, if I said an error bound, there's some *N* [*downward motion found in Figure **15***] where everything afterwards [*waving her hand to the right of where she did her downward gesture for N*] is going to be within that error bound, that's not going to make any sense.
Megan: It's not even going to work because-
Belinda: Because it's not going to work at all [*rightward gesture for error bound found in Figure **16***].
Craig: 'Cause it's-
Belinda: because right here I can set my error bound to be, like, that space apart [*repeats first picture in Figure **16***]. Then it's not going to work [*repeats similar gesture depicted in Figure **16***].

Based on her utterances and gestures, Belinda appeared to be directly overlaying the roles that ε and *N* play in two axis sequence and series graphs to their graph of Taylor series convergence for e^x. She correctly comes to the conclusion that in these roles, ε and *N* are "not going to work." Unfortunately, by applying ε and *N* in this way, she was neglecting the proper role of the index and the independent variable.

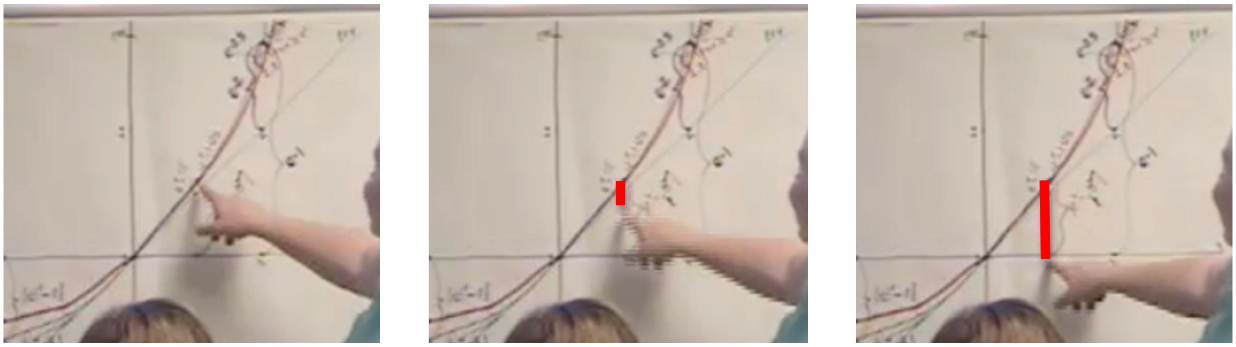


Figure 15. Belinda's Vertical Gesture of an "*N*" Value
*Red vertical "*N*" line added to illustrate what she was tracing.

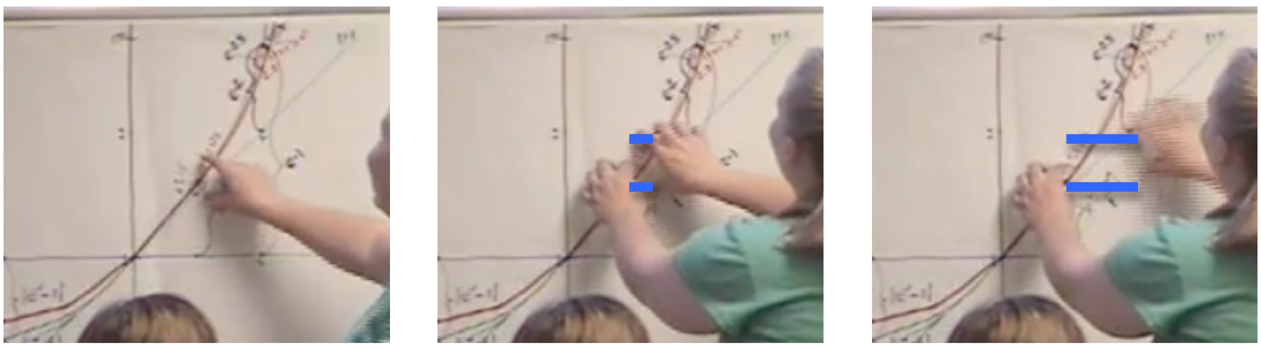


Figure 16. Belinda's Horizontal Gesture of an "Error Bound"
*Blue horizontal "error bound" lines added to illustrate what she was tracing.

Vertical number lines. In the calculus class students had depicted sequence and series graphs using not only the standard two axes graphs with axes for a_n and n, and S_n and n, respectively, they had also depicted sequence and series convergence on vertical number lines. *Figure 17* illustrates one such vertical number line for an increasing series converging to 5 drawn by Megan and Belinda. On the right of the number line, individual partial sums have been indicated using S_1, S_2, etc. and the terms composing the partial sum have been indicated as the difference between partial sums to the left of the number line using a_1, a_2, etc.

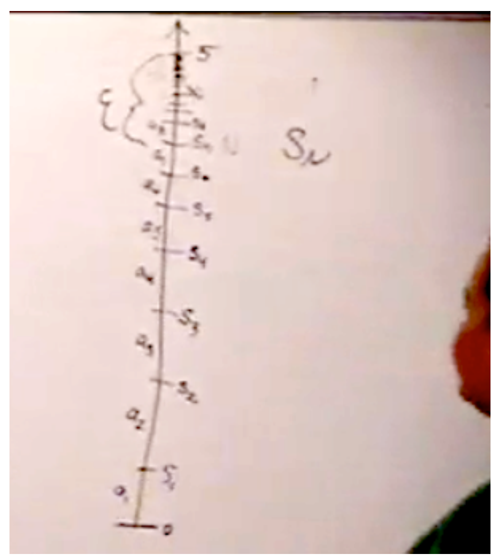


Figure 17. Vertical Number Line as Depicted by Megan and Belinda

Following a prompt from the facilitators to talk about their series definition for convergence in light of the vertical number line, the ε and N appearing in *Figure 17* (N appears very faintly in to the right of the number line) were added by Megan and Belinda while they were discussing components of their definition. Approximation terminology permeated their discussion as they consistently made references to approximations to 5 as the S_n partial sums, "errors" for approximations viewed as the difference between 5 and the corresponding approximation, "error bound" as ε, approximations "within" their error bound (the first such approximation denoted with index N), and how bounds on the error could be made small. From being given the prompt by Craig to producing a description of series convergence consistent with formal theory took the students around two minutes.

When the facilitators guided Megan and Belinda back to talking about Taylor series convergence, they then began to rearticulate their notion of convergence. During this rearticulation they both initially struggled in coordinating the roles of ε, N, and x. But then unprompted, they both drew graphs of functions resembling e^x with multiple vertical numbers lines illustrating multiple partial sum approximations for particular values of x. After another subsequent application of their series definition applied to one of these newly drawn vertical number lines appearing underneath the e^x like functions, Belinda suggested that they "expand that definition [*pointing to their series definition*] for all x-values that exist" because, as she would later say, "the Taylor series is essentially these number lines but there's just a whole lot of them."

Their final Taylor series convergence definitions. When they initially started to expand their definition, while reading both their series convergence and their prior start to a Taylor series convergence definitions, Megan suggests, "So we can say convergence to $f(x)$ for all x when for all epsilon greater than zero, dot, dot, dot." Even though Megan indicated putting the "for all x" at the beginning of their definition, Belinda suggested putting "for all x" at the end and the following definition was produced: "A Taylor series converges to $f(x)$ when $\forall \varepsilon > 0$ there exists some N such that $\forall n \geq N \ \left| f(x) - S_n \right| \leq \varepsilon$ for all x."

After more probing by the facilitators using their previously drawn vertical number lines, the students clearly articulated that given a fixed error bound, N is different for different values for x and the same N that works for a specific value of x also works for all those values of x closer to the center but not the other way around. Apparently building off of this idea, instead of moving "for all x" forward in their definition, they suggest that their N in this definition is an N that is not found until "all the way up to infinity [*Belinda waving right hand to the right of the graph*]" and that this N works for all values of x. For the students this appeared to be some sort of idealized N that corresponded to x at infinity and if so, since an N for a specific value of x works for all those values of x closer to the center, this N at infinity would work for all values of x. But this N, as Belinda admitted caused a problem because it "may be impossible to find."

After the students admitted the potentiality impossibility of finding such an N, the facilitators brought the students back to their articulation of their definition that included the "for all x" at the beginning. When they reconsidered putting the "for all x" at the beginning, they immediately stated:

Megan: I think if we put this up here it would be restricting the N to be within epsilon of the x that we've chosen.
Belinda: I see what you mean. So like, depending upon what x-value we're at-
Megan: Yeah.
Belinda: -we would have a new N.
Megan: Yeah, and that would make it so we're not worrying about N not working further out.
Belinda: Yeah, try to find some N that's impossible to find.

For them, moving the "for all x" to the beginning of the definition resolved the problem they identified as trying to "find some N that's impossible to find" because their initial "N" had to work for all values of x. After some more discussion, which include them talking about S_n's dependence upon *x*, they eventually produced their final definition: "A Taylor series converges to *f*(*x*) when $\forall x$, $\forall \varepsilon > 0$ there exists some *N* such that $\forall n \geq N$ $\left|f(x) - S_n(x)\right| \leq \varepsilon$."

It should be noted that following the production of this definition, the facilitators probed the students' understandings and the students described the roles for *x*, *ε, N*, and *n* consistent with formal theory. Even though they admitted that their last definition's placement of "for all *x*" causing consecutive universal quantifiers felt "goofy," they felt like their definition now best captured Taylor series convergence.

Conclusion and Discussion

It is remarkable that the students reinvented and unpacked the formal definition of series and pointwise convergence within such a short time. During the iterative refinement process for these definitions, the students faced multiple problems, whether the problems be problematic issues not directly identified by the student or problems directly identified by the student. One of the main problems that students faced during their reinvention activies of producing a series definition was the construction of series graphs. Problems were more numerous when the students were attempting to reinvent a Taylor series convergence definition. One of the more notable problematic issues was the students viewing the approximated function and Taylor polynomial as the same on some sort of growing interval of exactness that grew as the index of the Taylor polynomial increased. Some of the problems directly identified by the students after they were attempting to leverage their series convegence definition included how to handle the *N* in light of the independent variable and how to incorporate the "for all *x*" into their Taylor series convergence defintion.

We claim that the timely instructional guidance provided to the students during the teaching experiment successfully supported them engaging these challenges and their subsequent reinvention of these definitions. For example, the facilitators having students produce graphs of series and Taylor series convergence gave students a reference point for which they could refer to during the construction of their definitions. The guidance provided by the facilitors suggesting to the students that they produce series graphs without focusing on particular formulas freed the students from being constrained by attempting to recall specific series formulas and allowed them to produce more series graphs. The facilitators served as eventual conflict producers by having the students produce Taylor series formulas that led to their eventual recognition that the tail of the series had value and therefore, a Taylor polynomial and the approximated function in these cases could not be identical over an interval. The guidance to produce and unpack vertical number line graphs supported students in seeing the graphs of Taylor series as comprised of convergent series at each *x*-value where *N* is dependent upon *x* as well as *ε*. After students had wrestled with the problem of where to put the "for all *x*," the facilitators acted as solution providers and suggested that they reconsider their placement of "for all *x*" to best capture *N*'s dependence upon *x*.

We also claim that the prior activity of defining sequence convergence became a means for supporting the students' definitions of series and Taylor series convergence as they recognized similarities between sequences, series, and Taylor series convergence in the context of their graphs and their definitions. For example, problems such as addressing the meaning of "infinitely close," the quantification of the error bound, and multiple uses of the "*n*" notation, which had all been issues during the reinvention of a sequence definition (see Oehrtman et al., 2011), were non-issues during their reinvention of series and Taylor series convergence definitions. Furthermore, when presented with the task of defining series convergence, they immediately saw similarities and started connecting elements from their sequence convergence definition to their forming series convergence defintion. Additionally, these connections were strong as evidenced by them being able to produce a formal series convergence definition in a matter of a couple minutes. Likewise, it is through the students' reflection on the components of their series convergence definition in the context of vertical number lines, the role of vertical number lines in the context of Taylor series graphs, and their coordination of the "for all x" with their prior series definition that they came to eventually produce a pointwise convergence definition for Taylor series consistent with formal theory.

Furthermore, the students' emerging approximation scheme greatly supported the students in meaningfully recognizing similarities between definitions and interpreting components within a definition. First, the approximation terminology that they had learned from class allowed them to meaningfully interpret the role of approximations, error, and error bounds within a limit context. For example, in the context of series, this was evidenced by references to partial sums as approximations, $|U - a_n|$'s as errors, and ε's as error bounds together with corresponding graphical interpretations. Second, the approximation terminology had been abstracted by the students in such a way as to allow them to use this terminology to identify and coordinate similarities between structural elements across different limit contexts. For example, this was evidenced by their references to approximations as terms of a sequence, partial sums of a series, and Taylor polynomials. During the creation of their series convergence definition, the approximation terminology helped bridge the gap between similar components of their sequence definition and their emerging series definition. Therefore, their emerging approximation scheme helped to organize their series definition in light of their sequence definition. Likewise, this type of interplay between elements of their series definition and their emerging Taylor series convergence definition using approximation terminology can be seen during their creation of their Taylor series convergence definition. In essence, this demonstrates how the approximation framework can support students in the highest level of abstraction indicated in *Figure 9* and how this abstraction can further support students through reflection upon common structures and actions performed on relevant tasks in class and during reinvention.

As in Oehrtman et al. (2011), we acknowledge that individual students follow unique learning paths and that orchestrating the type of discussions needed for reinvention within an entire class will involve significant differences in what we have described here. Even so, this study demonstrates that reinvention of these definitions can occur and how it might occur. Furthermore, it demonstrates potential cognitive challenges, how challenges may be resolved, and how students can then leverage their resolution experiences to engage new challenges. We look forward to using these results to guide our future work to develop classroom activities for introductory analysis courses.

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