IMPROVING THE QUALITY OF PROOFS FOR PEDAGOGICAL PURPOSES: A QUANTITATIVE STUDY

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At last year's conference, we presented a qualitative study providing insight into what mathematicians believe makes a good proof for pedagogical purposes based on eight mathematicians' revisions of two proofs (see Lai & Weber, 2010). In this paper, we empirically test four hypotheses generated from last year's study. This year's study provides quantitative support for the claims that mathematicians believe (1) adding an introductory sentence stating the goals of the proof improves its pedagogical quality, (2) formatting key equations in a proof to emphasize their importance improves their pedagogical quality, and (3) unnecessary statements in a proof lowers its pedagogical quality.

Key words: Proof; Proof presentation; Undergraduate mathematics instruction

1. Introduction

1.1. Proof as explanation in collegiate mathematics education

Providing instructional explanations is a fundamental activity in mathematics instruction (Charalambous, Hill, & Ball, in press); explanation is not only a primary means by which teachers convey mathematical subject matter to their students (e.g., Leinhardt et al, 1991), but also a means of establishing classroom norms, illustrating productive metacognitive processes, and representing the discipline of mathematics (Larreamendy-Joerns & Muñoz, 2010; Schoenfeld, 2010). For these reasons, Charalambous, Hill, and Ball (in press) contend that "providing instructional explanations lies at the heart of teaching, for it requires transforming the content in mathematically legitimate *and* pedagogically appropriate ways" (authors' emphasis). There has been a great deal of research on instructional explanations, largely focusing on teachers' difficulties or inability to provide adequate explanations (e.g., Ball, 1988; Inoue, 2009; Leinhardt, 1989; Lo et al, 2004; Thompson & Thompson, 1994, 1996; Thanheiser, 2009), and more recently on improving teachers' abilities to provide high quality explanations (e.g., Charalambous, Hill, & Ball, in press; Kinach, 2002; Inoue, 2009; Thanheiser, 2010). However, most of this work has been done with elementary mathematics teachers and little work of this type has been done with teachers of tertiary mathematics.

In this paper, we focus on teachers' pedagogical practice in advanced mathematics courses at the tertiary level. In these courses, the predominant way of presenting mathematical subject matter to students is via mathematical proof. By mathematical proof, we mean "a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion" (Griffiths, 2000, p. 2). We note further that other characteristics of this genre include the use of precise definitions rather than informal descriptions of concepts, the (relative) lack of diagrams and other intuitive representations of concepts, and the use of logical syntax (e.g., Weber & Alcock, 2009).

Although many mathematics educators and mathematicians question whether these types of proofs are an appropriate way of conveying mathematics to students (e.g., Davis & Hersh, 1981; Hersh, 1993; Kline, 197; Thurston, 1994), there is little doubt that proof is the predominant way that mathematics is presented to students in advanced mathematics classrooms. For instance, Davis and Hersh (1981) asserted that "a typical lecture in advanced mathematics … consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation" (p. 151). Dreyfus (1991) claimed that although mathematics instructors may be aware that new mathematics is created through non-rigorous processes, this "does not usually prevent him or her from almost exclusively teaching the one very convenient and important aspect which has been described above, namely the polished formalism, which so often follows the sequence theorem-proof-application" (p. 27). Weber (2004) presented a detailed case study of one professor's teaching that was largely based on presenting proofs of theorems. Based on her observations of three mathematics professors, Mills (2011) found that these professors spent, on average, about half of their lecture time presenting proofs to students.

1.2. What makes a good pedagogical proof? Results from an exploratory study

In a previous paper, we investigated mathematicians' beliefs about what makes a good proof for pedagogical purposes by examining the ways in which eight mathematicians revised two proofs that were to be presented to undergraduate students (Lai & Weber, 2010). Our rationale for conducting this study is that the mathematicians presumably believed their revisions improved the proof for pedagogical purposes, either by removing an aspect of the proof that was undesirable or introducing a desirable feature into the proof. By analyzing what types of revisions that mathematicians made, as well as attending to their justifications for making these revisions, we could gain insight into what characteristics mathematicians believed made a good proof for pedagogical purposes and what mathematical and pedagogical reasons they had for these beliefs. Hence, through the analysis of our data, we believed we could form grounded hypotheses about what mathematicians valued in a proof and why. Four of the hypotheses that we generated our presented below:

(H1) Mathematicians believe that a proof may be more easily understood by undergraduates when it contains hypothesis and conclusion statements that make explicit what is being accomplished in the proof.
(H2) Mathematicians believe that emphasizing the main ideas used in a proof can improve its quality for pedagogical purposes.
(H3) Mathematicians believe that adding extra justifications to support an assertion can improve the clarity of a proof if that justification might be difficult for a student to infer on their own.
(H4) Mathematicians agree that including unnecessary, irrelevant computations or assumptions in a proof will lower its pedagogical quality.

We generated (H1) because all eight mathematicians in Lai and Weber (2010) added an introductory or concluding statements to at least one of the proofs. We generated (H2) because several mathematicians in Lai and Weber (2010) introduced formulas into the proofs they read because they felt these formulas represented the "key idea" of the proof or were "the heart of the matter". Further, four participants centered what they believed to be pivotal formulas in the proof to emphasize their importance. We generated (H3) because participants frequently added justifications to the proof if they felt students would have difficulty generating these justifications for themselves. We generated (H4) because one of the most common revision mathematicians in Lai and Weber (2010) made was removing statements that were not germane to the main ideas of the proof or were inferences that they believed would be trivial for undergraduates to make.

1.3. Research questions addressed this study

We believed the findings reported in Lai and Weber (2010) were interesting and potentially valuable. However, due to the qualitative and exploratory nature of our study, as well as the small sample size that we employed, we were obligated to be cautious about the generality of our findings. This is why we refer to the findings from Lai and Weber (2010) as hypotheses.

We believe that small-scale qualitative studies are indispensable in mathematics education, both for the development of theory and for the generation of hypotheses. However, we also contend that these types of studies should be the starting point of an investigation, not the ending point. In particular, we believe the generation of useful hypotheses is a crucial function of small-scale qualitative studies, but these hypotheses need to be rigorously tested to ensure their validity and generalizability. In this paper, we test hypotheses (H1), (H2), (H3), and (H4) in a large-scale quantitative study.

2. Related literature

2.1. The communicative functions of proof

A significant function of proof in mathematics is to provide conviction that an assertion is true (e.g., Harel & Sowder, 1998). However, in a seminal paper, de Villiers (1990) argued that proof is much more than that. To mathematicians, proof serves many functions beyond that of providing conviction. Proof can also be used as a tool to explain why a theorem is true, systematize a mathematical theory, or discover new theorems. de Villiers (1990) further argued that an important function of proof is communication. Providing mathematicians with a shared language and standards of argumentation facilitates debate about sophisticated mathematical ideas. Many educators have argued that proof should provide similar roles in the mathematics classroom; proof should not only be used to convince students that a theorem is true, but also to provide explanation and facilitate communication (e.g., Alibert & Thomas, 1991; Hanna, 1990; Healy & Hoyles, 2000; Knuth, 2002). These researchers also lament that proof often is not used for explanation and communication in these classrooms, consequently playing only a limited role in mathematics teaching.

2.2. Educational research on proof presentation

Although there have been few systematic studies on undergraduates' comprehension of proof, both mathematicians and mathematics educators have remarked that students generally find proofs confusing and learn little from reading them (e.g., Alcock, 2010b; Davis & Hersh, 1981; Hersh, 1993; Leron & Dubinsky, 1995; Porteous, 1986; Rowland, 2001; Thurston, 1994). Although there are many reasons why students might learn little from reading proofs, many argue that it is the linear and formal nature of proof that inhibits' students' understanding. Specifically, proofs often mask the intuitive representations that are needed to generate and understand them, and the jargon, logical syntax, and abstractness of a proof are intimidating barriers to comprehension (e.g., Hersh, 1993; Kline, 1977; Leron, 1983; Rowland, 2001; Thurston, 1994). Consequently mathematics educators propose different ways to present mathematical information to students other than formal proof (e.g., Alcock, 2009; Hersh, 1993;

Leron, 1983; Rowland, 2001), although we are not aware of any empirical studies that demonstrate the efficacy of these instructional suggestions. Aside from these instructional recommendations, research on proof presentation in mathematics education has been limited. In particular, we are not aware of any studies on what mathematicians believe are good proofs for pedagogical purposes or how mathematicians choose to present proofs to their students. The studies presented in this paper address these issues.

The primary aim of this paper is to investigate what mathematicians—the ones who usually teach advanced mathematics courses—think constitutes a good mathematical proof for pedagogical purposes. This fills two underrepresented areas of research in mathematics education. First, as Speer, Smith, and Hovarth (2010) argued, there is little research in mathematics education on how mathematicians actually teach. Second, based on a systematic review of the mathematics education literature on argumentation and proof, Mejia-Ramos and Inglis (2009) found that empirical research on proof has mainly focused on students' construction and evaluation of proof; their sample did not contain any empirical studies on how mathematical instructors (or for that matter, students) chose to present mathematical proofs. There are two reasons for undertaking this line of research. First, understanding mathematicians' beliefs about what constitutes a good proof for pedagogical purposes provides researchers with a better lens to interpret mathematicians' pedagogical behaviors, including their in-class behavior as well as how they prepare lecture notes for class. Second, as Alcock (2010b) noted, if mathematicians and mathematics educators are going to engage in a conversation about how to improve collegiate mathematics education, then it is necessary for mathematics educators to understand mathematicians' beliefs and values and to take these beliefs and values seriously. Mathematicians most likely will not adopt innovative instruction that is at variance with their beliefs and values.

3. Methods

3.1. Participants

We recruited mathematicians to participate in this study as follows. Thirty secretaries from mathematics departments in the United States were contacted and asked to distribute an e-mail to the mathematics faculty, post-docs, and Ph.D students of that department. These mathematics departments had strong national reputations within the United States. The e-mails that the secretaries sent to the recipients explained the study they were going to complete and invited them to visit a website with the study if they were interested. A total of 110 participants agreed to participate.

3.2. Validity of internet-based studies

We adopted an internet-based study to maximize our sample size using a methodology similar to Inglis and Mejia-Ramos (2009). The validity of internet-based experiments was studied by Krantz and Dalal (2000), who compared 20 internet-based studies with their laboratory equivalents and found a "remarkable degree of congruence" between the methodologies. A similar conclusion was reached by Gosling et al. (2004). Given these findings, and the impracticality of obtaining large samples of research-active mathematicians, we believe our methodology is justified.

3.3. Materials.

The materials used in this study are presented in the Appendix. Participants were shown a Master Proof . They were asked to compare this Master Proof to five other proofs (M1, M2, M3, M4, and M5) that had modifications to the Master Proof in blue. The participants were asked to judge whether these modifications increased or decreased the pedagogical value of the proof.

The first proof, M1, was used to test our first hypothesis, H1. It added an introductory and concluding sentence to the Master Proof to make explicit the proof framework being employed. M2 was used to test H2 by reformatting two important formulas in the proof to highlight their significance. M3 was used to test H3 by adding a justification in the proof that we felt might be difficult for students to infer. If H1, H2, and H3 were correct, we would expect mathematicians to view M1, M2, and M3 respectively as improvements to the proof. M4 and M5 were used to test H4. M4 added an irrelevant calculation to the proof, while M5 added an unnecessary assumption before applying the Mean Value Theorem. If H4 is correct, we would expect mathematicians to view M4 and M5 as lowering the pedagogical quality of the proof. We included two items, M4 and M5, to test the possibility that participants would view any additions or changes to the text positively.

3.4. Procedure

When participants visited the experiment website, they were first asked to indicate whether they were a Ph.D student, a post-doc, or mathematics faculty, as well as their level of experience. They were then shown the Master Proof and told that they would be shown modifications to the Master Proof; they would then be asked to judge whether the changes made the proof “more or less understandable to a second- or third-year undergraduate student”. Next, the participant was presented with a screen containing the Master Proof at the top of the screen and a modified proof beneath that. They were asked if the modified proof made the proof “significantly better”, “somewhat better”, “the proofs were the same”, “somewhat worse”, or “significantly worse”. We coded these responses as 2, 1, 0, -1, and -2 respectively. Along with their evaluations, the participants were given the option of commenting on their responses. For each modified proof, we performed an open coding of the participants’ responses in the same manner in which we coded participants revisions in the revision task. This process was repeated until the participants evaluated all five modified proofs. The order in which the modified proofs were presented was randomized by participant.

4. Results

A repeated measures ANOVA revealed a main effect on participants’ evaluations by which proof the participants evaluated ($F(4, 327) = 231.7$, $p<0.001$), indicating participants did not all believe the modifications were of equal quality. Of the 110 participants, 20 were mathematics faculty, 13 were post-docs, and 77 were Ph.D students. ANOVAs comparing the status and experience of the participants with their response patterns did not yield significant effects ($F < 1$, $p > 0.5$) in both cases, indicating that there was not a significant difference between how mathematics faculty members, post-docs, and Ph.D students performed on this task. Table 1 summarizes the main quantitative results from this study.

Table 1. Summary of data

Condition	Mean score	# participants who thought proof was better	# participants who thought proof was worse
M1	1.29*	97	4
M2	1.05*	88	2
M3	0.02	41	40
M4	-1.66*	6	98
M5	-1.12*	7	94

* Indicates a mean score statistically different than zero with $p<0.001$.

(Note participants who claimed the original and modified proofs had the same pedagogical value were not included as either participants who thought the proof was better or who thought the proof was worse).

4.1. M1: adding an introductory and concluding sentence

The results in Table 1 confirm H1—participants overwhelmingly believed that adding an introductory sentence that makes the framework of a proof explicit improves the pedagogical clarity of the proof. A summary of participants' comments is presented in Table 2. Of the 36 participants who evaluated the proof positively and volunteered to leave comments, 30 cited the benefit of giving students a "roadmap" to the proof and letting the students know explicitly what was being proved. Eight participants argued that the first sentence was good because it reminded the participants what the definition of injectivity was. These comments also reveal that some participants did not view the concluding sentence as an improvement, suggesting the main benefit of the changes lied in adding the introductory sentence.

Table 2. Comments from participants who evaluated M1 positively

Type of Response	Number of Responses (36 total)	Representative Responses
It is beneficial to state the objectives of the proof	30	"Stating what is to be proven is almost always helpful to students", "I think any proof benefits from a summary of what needs to be shown", "Roadmaps. Great."
The second sentence was less helpful than the first	5	"The last blue part is too much, I think", "The last sentence is superfluous"
It is useful to remind students of the definition of injectivity	8	"Yes! Repeat definitions like mad! Drill them into their heads!", "It's always nice to repeat the definition"

(*Note:* Some responses were assigned to more than one category).

4.2. M2: reformatting two important formulas

The results in Table 1 confirm H2—most participants believed that re-formatting the proof by placing significant equations in the proof on their own line and centered improved the pedagogical quality of the proof. Participants' comments confirm this finding. However, these comments also suggest another reason for why M2 was judged to be an improvement to the Master Proof: the additional spacing simply made the proof easier to read.

A summary of participants' comments is presented in Table 3. Of the 31 participants who evaluated the proof positively and volunteered to leave comments, 21 cited the benefits of making the proof less cluttered and easier to read, while ten cited the benefits of emphasizing the important equations of the proof. Whether some mathematicians believe that formatting *any* equation to increase spacing would increase the pedagogical value of a proof, or if this should only be done to important equations, is a question that can be addressed in future research.

4.3. M3: adding a justification in the proof

The results in Table 1 failed to support H3. As Table 4 illustrates, many participants (41) participants believed adding the extra justification improved the Master Proof. However, nearly an equal number of participants (40) felt the addition of the extra justification made the proof worse.

Table 3. Comments from participants who evaluated M2 positively

Type of Response	Number of Responses (31 total)	Representative Responses
The added spacing makes the proof less cramped and easier to read	21	"formulae on their own lines makes reading proofs easier", "Easier to read", "The original seemed cramped", "This is easier on the eyes, making the proof psychologically more appealing, I think"
The formatting draws the reader's attention to the important ideas of the proof.	10	"I think this makes the proof much clearer. Giving these equations their own line emphasizes that this is where the main points in the proof are happening", "Displayed equations draw the eye to important claims
Other	5	"Now the proof, once understood, can be recalled at a glance"

(*Note:* Some responses were assigned to more than one category).

Table 4. Comments from participants who evaluated M3 positively or negatively

Type of Response	Number of Responses (31 total)	Representative Responses
Favorable evaluation comments (N=10)		
Including the extra calculation may prevent potential confusion about the last step	8	“I can imagine a student being confused by the last step and this change would make it clearer”, “The new statement is redundant, but it saves time in thinking about why it works”
Other	2	“The extra explanation helps a bit, but it makes the proof Bulkier”
Unfavorable evaluation comments (N=15)		
There is benefit to having students make this inference on their own	2	“I decided it’s better for students to figure out the last step on their own”, “We should push students to complete this much reasoning on their own”
The audience is capable of making this calculation so it unnecessarily lengthens the proof	7	“Maybe ‘worse’ sounds harsh, but it adds more words and if you have proved the Mean Value Theorem, then a deduction like this should be straightforward”, “Clutter by repetition”
The added justification is not written well	6	“I think you should say f(x2) – f(1) is greater than zero, therefore f(x1) <> f(x2)”, “Not a good way to conclude the proof”
Other	2	“I suspect it takes more to read the inserted line than to realize what is being pointed out. It would take a study of actual undergrads to make sure”.

(*Note:* Some responses were assigned to more than one category).

A summary of the comments that participants left for both the positive and negative evaluations is presented in Table 4. Eight participants commented that they approved of the modification, believing the extra justification might prevent confusion. However seven who disapproved of the modification believed that the justification unnecessarily lengthened the proof; they argued that undergraduates should be able to make this inference on their own. This suggests that there may not be agreement amongst mathematicians as to what types of inferences undergraduates are capable of making. Further, these comments suggest a tension that mathematicians may feel when adding a justification; while adding a justification might make a proof clearer to some, it might also make the proof unnecessarily longer to others.

4.4. M4: adding an irrelevant calculation to the proof

The results in Table 1 confirm H4—participants overwhelmingly believed the inclusion of the extra calculation made the proof worse. A summary of the 40 comments left by participants who evaluated M4 negatively are presented in Table 5. Most of these participants noted that the included change made the proof longer unnecessarily, while 15 participants commented that the new equation was likely to be distracting or confusing for the reader.

Table 5. Comments from participants who evaluated M4 negatively

Type of Response	Number of Responses (40 total)	Representative Responses
The extra calculations unnecessarily lengthen the proof	31	"The additional formula makes it longer but not clearer", "That isn't the manipulation used at the end, why do it?", "The addition seems irrelevant"
Including the extra calculation will distract or confuse the student	15	"The blue material just adds another equation for the poor reader to parse", "Students may lose time and energy wondering why this line is there and/or develop misunderstandings"
The added expression divides by f'(3) before justifying that f'(3)≠0	4	"This comes before f'(x3) > 0 in the argument so you shouldn't divide through without indicating reasoning",

Other	3	"This takes the argument on a weird path"

(*Note:* Some responses were assigned to more than one category).

4.5. M5: adding an unnecessary assumption

Finally, the results of this study (presented in Table 1) also confirm H4—most participants viewed adding the assumption that f was a real-valued function in the proof diminished its pedagogical quality. A summary of the 40 comments left by participants who evaluated M4 negatively are presented in Table 6. Twenty of these comments note that adding this extra assumption could distract or confuse the students because it might lead them to try and make sense of why it was included.

Table 6. Comments from participants who evaluated M5 negatively

Type of Response	Number of Responses (40 total)	Representative Responses
The assumption is superfluous for this part of the proof	16	"The inserted detail is not pertinent to that area of the proof so I don't see any benefit to pointing it out", "This is unnecessary", "That's not the issue here, honestly"
Including this assumption is distracting and potentially confusing.	20	"The f was real-valued distracts from the main point", "This statement is distracting to the reader", "The blue phrase diverts attention away from the salient hypotheses"
It was implicitly understood throughout the proof that f was real-valued	21	"We are presumably referring to f being real-valued to appeal to properties of an ordered field. But in this context, it's quite obvious so the change just adds words", "The students most likely assume all functions are real-valued"

(*Note:* Some responses were assigned to more than one category).

5. Discussion

The results of this quantitative study revealed three things. First, the results confirmed H1, H2, and H4, supporting our claims that mathematicians believe proofs for undergraduates should make the proof frameworks explicit, format important equations to highlight their importance, and avoid adding unnecessary calculations and assumptions. Second, the comments left by participants deepened our understanding of why mathematicians valued these things. While adding introductory sentences to a proof can remind students of the meaning of definitions and provide a roadmap for how the proof will proceed, some participants found the concluding sentence to be unnecessary. Although some participants commented that the formatting of equations in M2 highlighted their importance and illustrated the main ideas of the proof, even more participants preferred the formatting because it made the proof less condensed and easier to read, suggesting that spacing, and more generally, visual appearance are important aspects that mathematicians value in proofs for pedagogical purposes. The comments on M4 and M5 illustrate one reason that mathematicians prefer that proofs do not have unnecessary calculations or assumptions. They believe these may distract the reader from the main ideas of the proof and confuse the reader by encouraging him or her to try to understand why they were included.

Finally, the results illustrated that H3—that mathematicians value adding an extra justification to a proof—was more nuanced than we believed. In a sense, the results of our quantitative study are consistent with the qualitative findings we found in Lai and Weber (2010).

In Lai and Weber (2010), several participants suggested adding justifications to a proof to bridge logical gaps that students might find challenging. In the current study, a sizeable minority of participants (41 out of 110 participants, or 37%) viewed the added justification in M3 to improve the pedagogical quality of the proof. However, to our surprise, a nearly equal number of participants believed adding this justification diminished the pedagogical quality of the proof, with some commenting that this inference would be obvious to the audience reading the proof. This suggests that mathematicians may strongly disagree on what level of detail is optimal in the proofs that they present to students, in part because they do not agree on what would be obvious to a student.

An interesting question is whether these mathematicians' beliefs are accurate. That is, do proofs that have the desirable features have concrete pedagogical benefits? For instance, will students who read such proofs understand them better or enjoy them more? These questions will be addressed in future research studies.

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APPENDIX: Materials used in this study

MASTER PROOF

Proposition. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.*

Proof. Let $x_1, x_2 \in \mathbb{R}$, where $x_2 > x_1$. The Mean Value Theorem implies there exists $x_3 \in [x_1, x_2]$ such that $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since, by hypothesis, $f'(x_3) > 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) > 0$. Therefore $f(x_2) \neq f(x_1)$. □

M1

Proposition. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.*

Proof. Let $x_1, x_2 \in \mathbb{R}$, where $x_2 > x_1$. To show f is injective, we must show that $f(x_1) \neq f(x_2)$. The Mean Value Theorem implies there exists $x_3 \in [x_1, x_2]$ such that $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since, by hypothesis, $f'(x_3) > 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) > 0$. Therefore $f(x_2) \neq f(x_1)$. It follows that f is injective. □

M2

Proposition. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.*

Proof. Let $x_1, x_2 \in \mathbb{R}$, where $x_2 > x_1$. The Mean Value Theorem implies there exists $x_3 \in [x_1, x_2]$ such that

$$f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since, by hypothesis, $f'(x_3) > 0$ and $x_2 - x_1 > 0$, then

$$f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) > 0.$$

Therefore $f(x_2) \neq f(x_1)$. □

M3

Proposition. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.*

Proof. Let $x_1, x_2 \in \mathbb{R}$, where $x_2 > x_1$. The Mean Value Theorem implies there exists $x_3 \in [x_1, x_2]$ such that $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since, by hypothesis, $f'(x_3) > 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) > 0$. As $f(x_2) - f(x_1) \neq 0$, it follows that $f(x_2) \neq f(x_1)$. □

M4

Proposition. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.*

Proof. Let $x_1, x_2 \in \mathbb{R}$, where $x_2 > x_1$. The Mean Value Theorem implies there exists $x_3 \in [x_1, x_2]$ such that $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since, by hypothesis, f is a real valued function, $f'(x_3) > 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) > 0$. Therefore $f(x_2) \neq f(x_1)$. □

M5

Proposition. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.*

Proof. Let $x_1, x_2 \in \mathbb{R}$, where $x_2 > x_1$. The Mean Value Theorem implies there exists $x_3 \in [x_1, x_2]$ such that $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$, so $x_2 - x_1 = \frac{f(x_2)-f(x_1)}{f'(x_3)}$. Since, by hypothesis, $f'(x_3) > 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) > 0$. Therefore $f(x_2) \neq f(x_1)$. □