SPANNING SET AND SPAN: AN ANALYSIS OF THE MENTAL CONSTRUCTIONS OF UNDERGRADUATE STUDENTS

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Abstract

We present a genetic decomposition using APOS Theory, about the way in which students can construct the concepts of spanning set and span in Linear Algebra. We also present empirical data coming from interviews made with 11 university students who had completed a course on Analytic Geometry which included an introduction to Linear Algebra. We report on our observations and suggest possible modifications to our initial theoretical analysis.

Key words: Spanning set, APOS Theory

Introduction

It has been reported in different research studies that learning Linear Algebra is difficult for university students and that many of these problems are related to the difficulties students face when learning abstract concepts (Dorier, 1997). Several studies have been carried out in different countries, and some authors have developed didactical proposals to teach specific concepts, such as vector space, linear transformations, and basis, or designed methodologies in order to help students to overcome the detected obstacles (Sierpinska, 1994; Alves Días& Artigue 1995; Harel, 1997).

In an earlier study about the construction of the concept of basis in Linear Algebra (Kú, Trigueros & Oktaç, 2008) we observed the difficulties that students have with the concept of spanning set and the coordination of the underlying process with the process related to linear independence. These difficulties seemed to interfere in a serious manner with the construction of an object conception of basis of a vector space. As a result we decided to carry out research in order to look at these concepts separately, so that we could offer an explanation about the construction of each concept and related problems.

Some literature published previously touch certain issues related to the learning of spanning sets focusing on task design, cognitive difficulties and suggestions for teaching (Nardi, 1997; Ball et al., 1998; Dorier et al., 2000; Rogalski, 2000). What we are interested in with this research is to offer a viable path that students may follow in order to construct this concept as well as explaining the nature of related difficulties while learning it. After completing our analysis of the empirical data, we also hope to make pedagogical suggestions.

Our research questions in this study are:

Do the constructions modeled in a genetic decomposition for the concepts of spanning sets and span useful in describing students' ways of dealing with problems related to these concepts?

What constructions can be associated to students' difficulties?

How can this information be used to make suggestions for the teaching of these concepts?

APOS Theory

APOS theory has been used successfully in explaining the construction of several concepts in undergraduate mathematics curriculum, and in designing theory based didactic approaches to teach them (Dubinsky, 1996). It has been proved to be effective for both purposes by means of studies concentrating in many different mathematical areas, such as Abstract Algebra, Differential and Integral Calculus, Statistics, and Discrete Mathematics (Dubinsky & McDonald, 2001). Its effectiveness is related to the possibility to explain certain processes that take place when learning advanced mathematics topics using the conceptual tools developed in the theory. Its use to study the construction of Linear Algebra concepts is more recent but results obtained so far are promising (Trigueros & Oktaç, 2005a; Roa-Fuentes & Oktaç, 2010; Parraguez & Oktaç; 2010; Trigueros, Oktaç & Manzanero, 2007). We continue with this line of research and use APOS theory to study the mental constructions and mechanisms involved in the learning of spanning sets.

APOS theory (Action, Process, Object, Schema) is an adaptation of Piaget's epistemological ideas to the learning of advanced mathematics at the university level. According to APOS theory an individual's mathematical knowledge and its development can be defined as:

"An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations" (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996).).

In this definition the main elements that enable a researcher to discern the way in which a student understands a mathematical concept, and that constitute the fundamental constructs of APOS theory are the mental structures called action, process, object and schema.

We say that students have an action conception of a mathematical concept if they perform transformations on objects in the form of step by step calculations or if they rely on memorized facts. According to Asiala et al. (1996) "an individual whose understanding of a transformation is limited to an action conception can carry out the transformation only by reacting to external cues that give precise details on what steps to take". Even if an action conception constitutes a limited form of understanding, the construction of action conceptions is crucial as a starting point to construct a concept.

After repeating actions on objects and reflecting upon these actions, students interiorize them into a process. Students who show a process conception are able to think about the transformations and describe them without a need to perform each step explicitly.

When students reflect on the processes and are able to think of them as a whole, we say that they have encapsulated the process of applying a transformation into an object. This implies that they are able to apply actions on the newly constructed objects. Students who show an object conception of a concept are also able to de-encapsulate an object into the process from which it originated.

Actions, processes, objects and other schema, and connections between them form what is called a schema in APOS theory. Therefore it can be said that a schema for a mathematical concept is a coherent collection of actions, processes, objects, and other schema that are related as a structure in an individual's mind and that can be used in a problematic situation related to a particular mathematical topic or concept. The coherence of the schema refers to the ability of the individual to decide whether it is possible to work on a mathematical situation using that schema (Dubinsky & Mc Donald, 2001).

According to APOS theory, the mechanism used in transitions from a type of conception to another is abstractive reflection. Trigueros (2005b) mentions that this mechanism is activated through the physical or mental actions that an individual makes on a knowledge object.

Transitions from action to process or from process to object do not occur in a linear way. Students can show action conceptions for some aspects of a given concept they are learning, and show, for example, a process conception of other aspects of the same concept. It can be even difficult to decide the kind of conception a student shows because it is possible to interpret her or his explanation for a specific problem where some elements can be regarded as actions or processes and others as processes or objects. Also it is important to clarify that students' responses to problematic situations are necessarily related to the cognitive demand required by the problem. If the solution of a problem only requires actions, students will certainly use actions to work on it; which does not mean that they cannot show another kind of conception in another problem situation.

When using APOS theory it is necessary to develop an idealized and detailed description of the actions, processes, objects, schemas and their relationships occurring in the construction of a mathematical concept. This model is known as a genetic decomposition of the concept in question. The viability of a genetic decomposition can be tested empirically using students' work. The results of the analysis of the data obtained from this source are used to refine the genetic decomposition so that it gives a better description of the way students construct that concept. A genetic decomposition can also be used as a guide in the design of teaching materials. It is important to clarify that several different genetic decompositions can exist for the same mathematical concept; what is important, however, is for any genetic decomposition to describe what is observed in students' work.

Based on this conceptual framework the research reported in this paper followed these steps: First we developed a possible genetic decomposition for the concept of spanning sets, afterwards we designed a research instrument to probe students' mental constructions and to test the viability of our genetic decomposition. We are in the process of analyzing students' responses to the questions in the instrument to describe the mental constructions that students have made relative to the concept of spanning sets. Now we present the work involved in each of these steps, commenting on some of the results that we obtained.

Preliminary Genetic Decomposition for the concept of spanning set

According to the methodology linked to the APOS theory "research begins with a theoretical analysis modeling the epistemology of the concept in question: what it means to understand the concept and how that understanding can be constructed by a learner. This initial analysis, marking the researchers' entry into the cycle of components of the framework, is based primarily on the researchers' understanding of the concept in question and on their experiences as learners and teachers of the concept" (Asiala, et al. 1996). It is worth mentioning that in our genetic decomposition we include the constructions we consider necessary for the students to differentiate the meaning of the two concepts: spanning sets and spanned space.

Prerequisites

One possible set of concepts to start the construction of the notions of spanning set and spanned space are vector space, variable and solution set for a system of equations.

- Vector space concept is fundamental in the construction of several other linear algebra concepts, in particular those of spanning set and spanned set. We consider that students need to be able to recognize some familiar vector spaces that they have worked with, such as spaces with dimension 1, 2 or 3 and with defined elements (such as n-tuples). Students must also recognize that there are sets containing elements other than numbers that can be considered as vector spaces, for example, polynomials and matrices, and to be able to work with them.

- We consider that the solution set of a system of equations plays an important role in the interpretation of the meaning of spanning set and spanned space, so we consider that the students should demonstrate an object conception of this concept. According to Trigueros et al. (2007), students demonstrate an object conception of this concept if they can represent in parametrical form the solution set of a given system or describe its geometric representation accurately.

- Variable is another concept we consider important to start the process of construction of the above mentioned concepts. We consider that students need to work with variables as mathematical objects and so they need to understand variables as unknowns, general numbers, or parameters, and variables in functional relationships, and move flexibly between all these representations (Trigueros and Ursini, 2003).

Mental Constructions

Given a vector space V, a specific set S of vectors from V and a specific scalar field K, students need to perform actions on the vectors of S and the scalars. These actions consist of performing scalar products and sums of vectors in order to obtain a new vector of V. Coordination of these actions is interiorized into the process of construction of a new vector which is an element of the vector space, that is, into the process of construction of linear combinations. This process also implies that the student can verify if a given vector can be written as a linear combination of a given set of vectors. We consider this as a process conception of linear combinations.

Through actions or processes on a given set of vectors S, students can verify if there are scalars in K that can be used to express the elements of a new set of vectors T in the vector space V as a linear combination of S. This process is coordinated with the process of finding the solution set of the resulting system of equations taking into account the notion of variable. The result of this coordination is the process of finding a set of scalars. Through the action of forming the linear combination with these scalars and S, the student can verify that S generates T. This process is generalized to include different instantiations of S and T. When the set S can be considered as a whole, whenever it is needed, this process has been encapsulated into an object that we may call spanning set.

The object conception implies that students are able to explain that a given vector space can be generated by different spanning sets, that these sets do not necessarily have the same number of elements, or common elements, and that the number of elements in general is not the only determining factor if a set is a spanning set for a vector space.

By reversing the last process students can construct the spanned set of vectors. They can also perform on it the actions needed to verify that this set T is a vector space or a subspace of a given vector space V. This process can be generalized to determine if V can be formed by all the possible linear combinations of the set of vectors T. This generalization process is encapsulated into a new object that can be called a generated set, or spanned set, by the given set of vectors. These constructions must enable students to differentiate the concepts of spanned space and spanning set.

It is important to point out that we are not ignoring those concepts that are related to spanning sets, such as linear independence or dependence, basis, dimension, etc. We emphasize that our analysis is focused on how the construction of the spanning set can help students understand other concepts related to it, or if there are difficulties in the construction of this concept that act as obstacles when relating it with other Linear Algebra concepts.

We are also aware that the construction process can be started by the construction of the spanned (generated) set and followed by the construction of the spanning set. Experimental data would be needed to compare these different construction processes.

Based on this genetic decomposition we can identify some observable behaviors which we can link to the described mental constructions that students might display when solving problems related to the concepts of spanning set and spanned space.

We will consider that students have an Action conception of spanning set and spanned set if they show a process conception of linear combination, can use it to verify if there are elements of K that can be used to write a specific vector from V as a linear combination of the vectors of the set S. When verifying if the set spans the given vector space the student can do it only with specific vectors.

If the student is able to generalize the previous actions and running through the elements of S in her/his mind considers that every vector of V can be expressed as a linear combination of the elements of the set S, we will say that this student displays a process conception of spanning sets.

A student will display an object conception if she or he demonstrates that he or she can apply actions on the spanning set or the spanned set. This implies, among other things, that he or she considers that the spanning set is not unique, that different spanning sets do not need to have common elements, and that the number of elements of a set by itself is not a valid criterion to determine if the set is or not a spanning set. It is also important that she or he considers the fact that the generated space is the result of all the possible linear combinations of the spanning set, and that these two concepts are different.

Methodological aspects

We then designed an interview that consisted in 7 questions, in order to test the viability of our genetic decomposition. This instrument was applied to a group of 11 undergraduate students who were taking an analytic geometry course at a Mexican university. This course consists of an introduction to Linear Algebra with a strong emphasis on the geometric interpretation and properties of vectors in R^n, but it also covers vector spaces with matrices and polynomials as elements. In our design of the interview questions we took into account different aspects of a spanning set. We asked questions of the type whether a certain set spans a given vector space, but we also asked the construction type of questions, namely given a vector space identifying possible spanning sets for it. We also asked the students to compare the vector spaces generated by different spanning sets. By dealing with different aspects of the concept of spanning set in this manner, we hope to shed light on where the difficulties lie and verify the related mental constructions.

Interviews were tape-recorded and all the written information was kept as part of the data set. These interviews were analyzed according to our theoretical framework. The analysis focused on the identification of the mental constructions used by students while working with the problems and their comparison with those modeled in the preliminary Genetic Decomposition. We also focused on how students related different mathematical concepts.

Results and Discussion

In order to illustrate some difficulties related to the construction of the concept 'spanning set', in this section we give some examples from the interviews that we conducted, interpreting the data from the theoretical lens of APOS theory. In the extracts below students are presented by pseudonyms and 'I' stands for the interviewer. The two interview questions we selected for this purpose are numbered (3) and (5).

3. Let W be the subspace of R^3 that consists of all the points (x, y, z) such that $x+3y-4z=0$. Find a spanning set for W that contains two vectors. Can only one vector span W? Can three vectors span W? Justify your answer.

Question 3 has the purpose of identifying the strategies that students use, in determining a spanning set for a plane given by an equation. We also want to find out if students can make connections between the number of vectors in a set and its spanning properties.

We start with Javier, who writes the following after reading question (3) and comments on it:

W subespacio de $\mathbb{R}^3$ (x,y,z)

$x+3y-4z=0$ $(1,3,-4)$

Javier: I think it can be written this way, the x, the value of y, the value of z, and I think I can put it like this (writes):

$$\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$$

Now I would have to think how to span it, and they are asking me to span it with two vectors, right? So I think it can be put as $x=4z-3y$… (He writes $\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = a\begin{pmatrix} \end{pmatrix} + b\begin{pmatrix} \end{pmatrix}$). This (he points to the empty parenthesis multiplied by *a*) times *a* and this (writing the other empty parenthesis multiplied by *b*) times *b*.

Javier: This one (referring to the first vector), I want to decompose it in two so that when we do it, after we multiply it by this (referring to *a*) and by this (referring to *b*) and then we add them, it gives me the subspace of R^3.

When the interviewer asks him how he can check if the set he proposes really spans W, Javier does the following calculation and declares that he's done.

$$a\begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} + b\begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4a-3b \\ 3b \\ -4a \end{pmatrix} \quad \begin{matrix} b=y \\ a=z \end{matrix}$$

$$\begin{matrix} 4z-3y = x \\ 3y \\ -4z \end{matrix}$$

The interviewer asks again how he knows that the two vectors span W and he replies by saying that the two vectors are linearly independent. When the interviewer insists on the question of how to check the spanning of W by these two vectors, Javier says he doesn't know how to do it.

In the extracts shown above, we can observe that Javier chooses a vector with which to work, that is the normal vector of the given plane. Possibly the fact that a plane is determined by its normal vector (fact that is emphasized in the Analytic Geometry course he had been taking) makes him believe that it can be chosen as the representative of the given plane and that by working with it he would be working with the plane. Then he writes the normal vector as a linear combination of two other vectors. The answer given by Javier shows that there is a lack of coordination between the processes of belonging to a set and forming linear combinations, which is necessary in order to have a process conception of spanning sets. Javier, by only working with one of these processes, namely forming linear combinations, and not coordinating it with the other one, is not able to find a spanning set: the two vectors he proposes do not belong to the subspace in question. Afterwards, Javier comments on the possibility of spanning the given subspace by only one vector, or three vectors. However, obviously he is working with another subspace, the one generated by the normal vector.

> Javier: Then I will go to … if only one vector spans W, and I think yes, because since it is a subspace actually it is not the whole of R^3, and since it's only a… I think it's only a line, so I think it can be spanned by only one vector. And I think the third one is 'can three vectors span W?'. Yes, but they won't be linearly independent, since if they were linearly independent they would span all of R^3… I think.

Implicit in his answer above is the assumption that if a vector v_1 can be written as a linear combination of two other vectors v_2 and v_3, the subspaces generated by the two sets $\{v_1\}$ and $\{v_2, v_3\}$ are the same. Furthermore, it seems that Javier only makes a difference, in terms of dimension, between two kinds of subspaces: the vector space itself and all other proper subspaces, these latter ones all being put into the same category.

Then when he is asked to describe what it is that makes a spanning set, the following interchange occurs:

> Javier: By means of combinations of the elements it can generate a space.
>
> I: Ok, so how could you check whether that really is a spanning set for W, taking into account what you have just said?
>
> Javier: Well I am thinking in another characteristic because I see that these are linearly independent and combining them it comes out but there has to be another characteristic in order to determine if with that it will be a spanning set.
>
> I: So you make the linear combinations, and what does that imply?
>
> Javier: I make the linear combinations and it gives me the set.
>
> I: The set. Which set?
>
> Javier: The generated set.

One of the difficulties Javier is experiencing has to do with a confusion between two types of tasks: generating a subspace by using a set of given vectors, and given a subspace finding a possible set of spanning vectors. This and other data imply that a student might have different conceptions regarding the two concepts spanning set and generated subspace, and the related constructions may not be connected in the student's mind.

Now we pass to Oscar who makes use of the normal vector as well when he starts working on the problem, but uses a different strategy to proceed.

Oscar: (writes)

$$x+3y-4z=0$$
$$N=(1,3,-4)$$
$$n_{\perp}= (4,0,1) \quad (1,3,-4)\bullet(4,0,1)=0 \quad \text{and} \quad P\ (5,1,-2)$$

So here what I did was, I have the equation of a plane in R^3, I find the normal (vector), then I find the perpendicular to the plane, well making sure it gives zero. When I have it I look for a point that passes through the plane and well with that I can express the equation of the plane (writes):

$$\Pi = \{P|P= (5, 1,-2)+ t(4, 0,1)\}$$

But I need another… a point of the plane, the normal, I need another one. With only one vector I cannot span W.

Oscar: Because in the equation of the plane I have two parameters so well, I have any point and two parameters, right? Well two direction vectors and parameters in the reals. So... but… in order to span W which is an equation of the plane in R^3, I always need two vectors. So with one I would be spanning a line.

I: Ok

Oscar: And with three vectors I could be sp... if they are linearly independent I could be spanning R^3 and if they are not linearly independent no… Supposedly it spans a plane but if they are not linearly independent I cannot make sure that they span. But with two vectors I can span W.

At this point the interviewer goes back to the first problem and asks how he would obtain the spanning set that is being asked for. Oscar responds by saying that he doesn't remember. When he is asked what is required for obtaining a spanning set, Oscar gives the following answer:

Oscar: Being linearly independent and belonging to the set and being linearly independent in order to span it.

Later he mentions that the set {(5, 1,-2), (4, 0,1)} could be a spanning set, but that he would have to verify linear independence. He writes a system of linear equations and by solving it he affirms that the set is linearly independent and hence a spanning set for W. Although the vector (5,1,-2) does not lie on W because of the negative sign, it can be observed that Oscar is beginning to coordinate the two processes that give rise to the process of spanning a subspace. However in his interview too we observe that he has different types of conceptions related to the two concepts 'spanning set' and 'spanned subspace'.

5. Let $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and let $H = \left\{\begin{bmatrix}s\\s\\0\end{bmatrix} s \text{ in } R\right\}$. Therefore each vector of H is a linear

combination of $\{v_1, v_2\}$, since

$$\begin{bmatrix}s\\s\\o\end{bmatrix} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Answer the following question: Is $\{v_1, v_2\}$ a spanning set for H? Justify your answer.

With question (5) we emphasized explicitly the belonging of the vectors of a spanning set to the subspace that they generate.

Pedro gives the following response to this question:

> Pedro: They are giving me these two vectors and the set of elements. Well, the set will be all the elements that are in this form, where their first two elements are equal (s,s,0), so a linear combination of v_1 and v_2 will give you all of H, but not the combination of v_1 and v_2; that is there will be things that they generate that will not be in H. They will generate all of H, but also more things.
>
> I: So is it a spanning set for H or not?
>
> Pedro: No, it is not because they will generate even more. Yeah, I mean the linear combination of these two, I mean the span of v_1, v_2 will be greater than the set H.
>
> I: And what could be a spanning set for H?
>
> Pedro: A set… it could be for example the vector (1, 1,0), that is the set H is a set of dimension 1. You only need… you can only have one and that will span everything.

As the above extract shows, Pedro does not base himself only on linear combinations. Although he doesn't make explicit reference to belonging of the vectors to the subspace, he employs the idea that the linear combinations formed by the vectors of a spanning set should exactly generate the given subspace, and not more.

On the other hand, Carlos has difficulties in deciding whether the given set of vectors form a spanning set for H.

> Carlos: (writes)
>
> "Yes. All possible linear combinations of v_1 and v_2 span H. Neither v_1 nor v_2 can generate the third element, but H doesn't have it either, so it is not necessary."
>
> (then he explains)
>
> It's a spanning set for H because if we take all the possible linear combinations in the reals, then clearly we can see that it can be any number… well, any number in H and for example H doesn't have… it has a zero in the third element so, no, well…it would be, it's not needed and we see it here… I mean none of the two has it so… If H had another *s* here for example (he refers to the vector (s,s,0)), it wouldn't be a spanning set for H, we would need another vector, which had for example, I don't know. If it were linearly independent and if it had an element in the last position, but since these two don't have it, but H doesn't either, then H can be spanned by these two vectors.

Although Carlos mentions "all possible linear combinations", he doesn't make any reference to the belonging of the vectors v_1 and v_2 to H. The interview continues with the following question, in order to provoke a reflection in Carlos:

> I: Can you give another spanning set for H?
>
> Carlos: (writes)

$$gen = \left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \right\}$$

I: Let's see. Why is this a spanning set?

Carlos: Because they are two linearly independent vectors and if we take any numb… I mean this is in reals, so if we take any number in H, well for example in H, I don't know, for it to be 1 and 1. We multiply this one by 1/5 added to this one multiplied by 1/3 and it spans H.

We can observe that Carlos still wants to obtain the vectors v_1 y v_2 in order to find a different spanning set. When the interviewer asks how he would find a spanning set for H if the question didn't give the set $\{v_1, v_2\}$, Carlos responds as follows:

Carlos: If I didn't have this? Well, *s* is in the reals, so it could be any number. Well, it would be enough to take two vectors that, I mean, with which I can generate a real number in the first one and a real number in the second one and that would be enough.

As we can see Carlos keeps thinking about a similar set to the one given in the question in order to span H, and does not bring into the situation the necessity of the vectors belonging to H.

Conclusions

In this paper we propose a genetic decomposition that models how students might construct the concepts of spanning set and span. The empirical evidence presented illustrates certain difficulties related to this construction. Preliminary results show that in order to find a spanning set of a given vector space, a process conception of the related concept is needed, since the task requires the coordination of two underlying processes: belonging to a set and linear combinations. Students tend to ignore the belonging part when working on situations involving the concept of spanning set.

Students who have a process conception of the concept of spanning sets can relate it to other concepts such as linear independence and dependence, and dimension. We also observe that in general it is easier for students to decide whether a given set spans a given vector space than constructing a spanning set for a given vector space. These data inform our theoretical analysis and we propose some modifications in our genetic decomposition to reflect these results. On the one hand there should be an explicit mention of two processes that need to be coordinated as mentioned in the paragraph above, and on the other hand it should be emphasized that the two concepts spanning set and span may be evolving conceptions that involve different types of constructions.

Another observation is that even mathematically speaking the coordination of the two processes spanning and being spanned seems to be simultaneous, cognitively speaking it may not be so.

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