Student approaches and difficulties in understanding and use of vectors

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A configuration for analyzing vector representations based on multiple representations, semiotic representation, cognitive development, and mathematical conceptualization, to serve as a new unifying framework for studying undergraduate student approaches and difficulties in understanding and use of vectors is proposed. Using this configuration, the study will explore five important transitions: physics to mathematics, arithmetic to algebraic, analytic to synthetic, geometric to symbolic, concrete to abstract, and corresponding student difficulties along epistemological and ontological axes. As a part of validation of the framework, a mini-study on undergraduate students' approaches and difficulties in understanding and use of vectors is introduced, and we see how useful this new framework is to describe and analyze student approaches and difficulties in understanding and use of vectors.

Key words: representation, vector, vector representation

Vectors are very useful tool for solving real world questions mathematically. Vectors are applied widely to various fields in natural science, engineering and mathematics, even in social science and economics. They also have a valuable role in mathematics itself regardless of any relationship to real world and still have its own significance in advanced mathematics. Undergraduate students usually experienced vectors in school physics and school mathematics. When students study undergraduate mathematics, they see vectors again in multivariable calculus, linear algebra, abstract algebra and geometry courses. Some students see vectors in introductory physics or engineering courses while they are studying vectors in mathematics. Although undergraduate students' experiences with the concept of a vector varied, students still have difficulties in understanding and use of vectors.

In this research, we explore the following: (1) constructing a framework to analyze student approaches and difficulties in understanding and using of vectors, (2) classifying approaches and difficulties, (3) seeing how much one approach prevails over the others in student thinking, and (4) looking for the partial sources of student difficulties.

Root of the Framework for Vector Representations

Most of the studies about multiple representations are centered at the concept of a function (Janvier, 1987; NCTM, 2000). In a case of functions, there could be verbal, table, graphical, symbolic representations and they are usually not hierarchical. Moreover, one can solve one problem with various representations. For example, finding x-intercepts of a function can be done with a table, a graph, a symbolic form, even with verbal representation. Having a question of functions, the use of multiple representations is worth to try because sometimes a different representation gives a different and new perspective to look at the problem with and to look for the answer that can help students' further understanding. Unlike the representations of a function, vector representations have a hierarchy and are strongly dependent on the contexts of given questions. To grasp what student approaches and difficulties are in understanding and use of vector representations, many different contexts and different levels of sophistication should be considered.

Vector representations have a strong hierarchy so that one cannot fully enjoy the privileges and the benefits of multiple representations of a vector until understanding enough levels of sophistication. Representations of a vector are strongly dependent on the contexts of given questions. If a question asks a proof of synthetic geometry, only a few representations can be applied.

Many mathematics teachers and professors already knew student difficulties from their experience of teaching. However, those difficulties are not classified systematically and they are very scattered and isolated. As Tall (1992) mentioned, “the idea of looking for difficulties, then teaching to reduce or avoid them, is a somewhat negative metaphor for education. It is a physician metaphor - look for the illness and try to cure it. Far better is a positive attitude developing a theory of cognitive development aimed at an improved form of learning.” To have more positive attitude, we need to have deeper understanding of student approaches and difficulties on vector representations to the level of the theory of cognitive development.

This necessity of reflecting cognitive development in representations arises in two different directions of study. According to Duval (2006), a simple definition of representation such as “something that stands for something else” can be interpreted from two different perspectives: mental representation theory centered at internal, external communications and mathematical knowledge acquisition; semiotic representation theory that focus on signs and their associations produced according to rules. However, Duval (2006) tried to combine them together and studied semiotic representations at the level of mind’s structure. His identified sources of incomprehension from semiotic systems of representation that have three different purposes: to designate mathematical object, to communicate, and to work on/with mathematical object. He insisted that a transformation from a sign to the other, or a substitution of some semiotic representations for another should be considered with three components of semiotic representation system: representation content, semiotic register used, and represented object. The difference between physics representation and mathematics representation of vectors is gleaned from these components.

Most studies about vectors are from physics point of views related with physical quantities and by physics educators. J. Aguirre and Erickson (1984); J. M. Aguirre (1988); Knight (1995); Nguyen and Meltzer (2003) are just a few of them. Some studies such as Watson and Tall (2002); Watson, Spyrou, and Tall (2003) attempted to analyze student approaches and difficulties on vector representations with more mathematical point of views. However, their studies cover only secondary level mathematics and the transition from physical thinking to mathematical thinking. This brings up a necessity of the new framework for investigating vector concepts that can cover vectors in more advanced and wider ranges of undergraduate mathematics as well as in physics and secondary level mathematics.

Student approaches and difficulties in learning and using of vectors in undergraduate mathematics are very complex issues that have not yet definitely resolved. Dorier (2002) brought up these issues and analyzed them with a series of research. However this book placed the focus at linear algebra so that vectors in geometry or other subject fields in mathematics were covered very briefly. Linear algebra is just one of the fields that requires the concept of vectors frequently, but most studies on the concept of a vector so far are regarded as parts of linear algebra (Dorier, 2002; Harel, 1989; Dorier & Sierpinska, 2001).

Lesh, Post, and Behr (1987) pointed out five outer representations including real world object representation, concrete representation, arithmetic symbol representation, spoken-language representation and picture or graphic representation. Among them, the last three are more abstract and at a higher level of representations for mathematical problem solving (Johnson, 1998; Kaput, 1987). However, in most cases, picture representation is not geometric enough to show geometric structures, and graphical representation does not reflect synthetic geometry point of views but rather reflect analytic geometry point of views.

The problem of vector representations lies not only on the multiple representations but also on the translations. Sfard and Thompson (1994); Yerushalmy (1997) are based on the assumption that students ability to understand mathematical concepts depends on their ability to make translations among several modes of representations. Tall, Thomas, Davis, Gray, and Simpson (1999) analyzed several theories of these. These translations or transitions are referred to as *encapsulation* by Dubinsky (1991) and *reification* by Sfard (1991). The work of Sfard (1991) appears to view both operational and structural conceptions as important in mathematical understanding. Sfard focused on an operational/structural duality of mathematical conceptions. A structural conception enables recognition (at a glance) and manipulation as a whole; an operational conception is grounded in actions, processes, and algorithms. Sfard contended that the development of students conceptions can be viewed as occurring in three stages; interiorization, condensation, and reification; and that "to expect that a person would arrive at a structural conception without previous operational understanding seems... unreasonable" (Sfard, 1991) The proposed configuration of vector representation reflects this idea of encapsulation or reification not just in symbolic modes of representation but also in geometric modes of representation that has not been studied much along with algebra view point (Meissner, 2001b, 2001a; Meissner, Tall, et al., 2006). One important progress is that this configuration of vector representations can capture the whole picture of encapsulation or reification both happen in a symbolic way and a geometric way simultaneously.

Construction of a Framework

This research focuses on issues arising when representations for vectors are utilized in undergraduate mathematics instruction. Ultimately, the issues we plan to investigate in the future include the following:

1. What student difficulties in understanding and use of vector representations can be identified in the undergraduate mathematics curricula?
2. What generalizations might be possible regarding the relative degree of difficulty of various representations in learning vector concepts? That is, given a class and content do some forms of commonly used representations generate a large number of difficulties?

To investigate the above issues, we propose a new framework combining multiple representations, semiotic representation, cognitive development and mathematical conceptualization in order to understand vector representations. This framework serves as a new unifying framework for studying student approaches and difficulties in understanding and use of vectors. Using this configuration, the study explores five important transitions: (A) physics to mathematics, (B) arithmetic to algebraic, (C) analytic to synthetic, (D) geometric to symbolic, (E) concrete to abstract, and corresponding student difficulties along epistemological and ontological axes. As Zandieh (2000) stated in her study on the framework for the concept of a function, "The framework is not meant to explain how or why students learn as they do, nor to predict a learning trajectory. Rather the framework is a 'map of the territory,' a tool of a certain grain size that we, as teachers, researchers and curriculum developers, can yield as we organize our thinking about teaching and learning the concept..." this new framework serves as a 'map of the territory'.

Introduction of the Configuration

Figure 1 is the configuration for analysis of student approaches and difficulties in understanding and use of vectors. It has two axes, ontological axis and epistemological axis. Ontology is the philosophical study of the nature of being, existence or reality in general, as well as the basic categories of being and their relations. Epistemology is the branch of philosophy concerned with the nature and scope (limitations) of knowledge. Vector representations can be classified into those two aspects. For example, geometric or symbolic representations of vectors are the matters of existence or being. Concrete vs. abstract characteristic is also related with ontological aspect of vectors. On the contrary, arithmetic to algebraic transition and analytic to synthetic transition are more related with epistemological aspect. Both ontological axis and epistemological axis provide a good way to locate and see a vector as a mathematical object on undergraduate mathematics.

Figure 1. Configuration of vector representations

The first quadrant of the configuration is related with mathematics. The third quadrant is more related with physics. The origin represents two important jumps from physics to mathematics. Ontologically, the origin is a shift of a view, from vectors as representations of physical quantities with physical units to vectors as representations of mathematical objects. Epistemologically, this origin is a shift related with understanding of mathematical equivalence relations. In the domain of mathematics, each axis has two important jumps that can be easily identified. Those jumps will be explored in the next sections. Ten different representations of vectors are identified in this configuration.

Arrows on a grid. This is a representation of a vector in physics. In physics, a vector is a quantity with direction and magnitude. To represent a vector, they usually use an arrow. In this representation, physical directions and physical magnitudes are important. Without Cartesian axes, physical directions, for example NSEW, can be used and physical magnitudes can be corresponding to the lengths of the arrows with some physical units that can be measured by a grid. Therefore measuring the length of the arrow is significant in this representation. Arrows could be on a plane or on a space with axes as well. However, in physics representation, we see objects or points with vectors as things that are not actually on a plane or on a 3D space, but are within a plane or within a space moving within a plane or within a space.

Arrows with axes and scale. This is the first representation of a vector in mathematics, closest to a physics representation. Once an arrow is put on the Cartesian coordinates, it loses its physical attachments such as physical directions or physical magnitudes. Instead, the direction of a vector can be described by the origin and a point away from the origin measured by the scales on each axis. The length of the arrow defines the magnitude. It does not have any physical units. We see the points or object as things on a plane or a 3D space.

Arrows with axes but no scale. This representation is very similar to the representation above. The only difference is that the exact length of an arrow is not important here but the relative length is important. Measuring the length of a vector at this stage is useless. However the role of the origin as a starting point of position vectors is still important. At this stage, even though a geometric object such as a triangle or a polygon lies on the coordinate plane, it will be interpreted as a collection of points.

Arrows without axes or scale. Because there are no axes and the origin at this stage, the direction and the magnitude of a vector are not critical issues. Instead, the structure of how several vectors are related is important. For example, three vectors on a triangle can produce a special structure of sum or difference of vectors. Angles between vectors, parallel, perpendicularity are essential features at this stage.

Numerical column/coordinate form. Once a basis is given, the tip of a mathematical position vector can give a set of numbers that specify the vector. That set of numbers can be described as a column or coordinate form depending on whether it is from the standard basis or not. At this stage, a vector lives in two or three-dimensional real vector space with an inner product.

Column/coordinate form with variables. Instead of a set of numbers, we use two or three variables and functions. For example, $\begin{pmatrix} a \\ b \end{pmatrix}$, (x,y), $(f(x),g(x),h(x))$. This form of representation is very useful for describing parametrized curves or surfaces in multivariable calculus.

Reduced symbols. At this stage, a single letter or two can represent a column vector with numbers or variables as well as an arrow. For example, $\overrightarrow{AB}$, or $\vec{u}$. It makes the structure of operations of vectors easier to be seen, but it is meaningless to calculate actual results with numbers. The length of a vector exists only in a symbolic form, not given in numerical values. However there is still strong connection to geometric representations so that the length is related with the actual geometric length.

Numerical n-tuple. This is just a four or above dimensional extension of numerical column/coordinate form. Because it is hard to connect a vector with an arrow with axes and scale, it can be viewed as abstract form. However, if one gives up the visual orthogonality of basis vectors, one can still possibly connect arrow to a vector in an abstract way. And at this stage, one can actually calculate the length of a vector by simple extension of the way measuring the length of a vector in two or three-dimensional real vector space with simple extension of Pythagorean theorem. This length is more abstract and constructed by the definition of the inner product of a vector space.

N-tuple with variables. Using variable entries, we can easily extend numerical n-tuple to n-tuple with variables.

Abstract elements in a vector space. This representation does not have any information of a single vector. It is used to describe the structure of an abstract vector space. Zero vectors, inverse vectors are defined in an abstract way. Axioms define a vector space.

Development of the Configuration

Observing the way the concept of vectors is described in high school textbooks, college textbooks and various research; observing and listening to the way that mathematics education researchers, mathematicians, graduate students in mathematics and mathematics education, and undergraduate students describe the concept of vectors were how this framework developed.

According to Hillel (2002), undergraduate linear algebra courses generally included three modes of description of the basic objects and operations. These three modes of description co-exist, were sometimes interchangeable, but are not equivalent. They were: the abstract mode using the language and concepts of the general formalized theory, the algebraic mode using the language and concepts of the more specific theory of $\Re^n$, the geometric mode using the language and concept of 2- and 3-space. Using these three description modes, Hillel (2002) classified the representations of a vector. In the abstract mode, a vector is an element in a vector space defined with axioms. In the algebraic mode, a vector was an n-tuple of numbers in $\Re^n$. In the geometric mode, a vector was a directed line segment, or point. Within each mode, vectors, vector operations and transformations had particular depictions, terminology and notation, and there were mechanisms that enable one to move from one mode to another. Hillel (2002) also subdivided the geometric mode into three levels such as coordinate-free geometry level, coordinate geometry level, and vector-as-point level. Sierpinska (2002) also classified students' thinking and reasoning in linear algebra courses into three modes. Synthetic-geometric, analytic-arithmetic, and analytic-structural modes of thinking and reasoning were those. This classification was different from Hillel (2002)'s classification in a way that this was more focused on students' thinking and reasoning than the descriptions from history or epistemology. Even if a student was working on vectors in $\Re^n$ that were the algebraic mode of description, he or she could use many different modes of thinking and reasoning to figure out the problem, the situation, and the solution. Each mode was not independent from the others. They co-existed and sometimes were interwoven with each other.

These classifications of the concept of a vector were very useful to understand what vectors were in mathematics, but not helpful to see the relationship among representations, the translations and what student difficulties were in learning the concept of a vector and use the vectors in various contexts. Pvalopoulou's work (as cited in Dorier, 2002) as an application and verification of Duval's theory of semiotic representation in the context of teaching linear algebra covered this relationships. The author distinguishes between three registers of semiotic representation: the graphical register (arrows), the table register (columns of coordinates), and the symbolic writing register (axiomatic theory of vector spaces). She proposed problems in one register and asked for translations into another imposed register. This brings out the asymmetric direction of the conversion activity: 7% success in converting from the table register toward the symbolic register, 72% for the opposite conversion, 5% success in converting from the graphical register toward the symbolic register, 40% for the opposite conversion, 83% success in converting from the 2D table register toward the graphical register, 34% for the opposite conversion, and 35% success in converting from the 3D table register toward the graphical register, 68% for the opposite conversion. These results gave warrants for the structure of the configuration and the existence of the levels of conceptualization in the configuration despite the graphical register did not cover the full geometric representations on the configuration.

These studies set up their goal as understanding vectors for linear algebra, so that the important part of vector use in geometry was somewhat neglected. Some students entered a first course in undergraduate geometry with a significant amount of previous experience of vectors in mathematics courses such as multivariable calculus or linear algebra where the representations of vectors were often symbolic. Geometric representations were not developed well enough in those courses. This may well serves as a stumbling block to use the geometric representations of vectors as a way to understand geometry. As a result, students may implicitly believe that use of vectors in geometry is less than desirable or vector geometry is just related with analytic (coordinate) geometry not related with synthetic (coordinate-free) geometry. This inclined scaffolding of vector representations should be overcome in both teaching and learning of vector concepts. Therefore, we needed to organize and structure a wide range of possible understanding and use of vector representations covering beyond linear algebra as levels of conceptualization of vectors across all the fields of mathematics.

Features of the Framework

This new framework has some important features. First, it suggests that the interplay between ontological aspect and epistemological aspect is critical in understanding and use of vectors and the key transitions between representations require both ontological and epistemological aspects of understanding simultaneously. Second, it can distinguish and put greater emphasis on difference between analytic geometric representations of vectors and synthetic geometric representations of vectors. It can also distinguish and put greater emphasis on difference between physical representations of vectors and mathematical representations of vectors. Furthermore, it distinguishes, shows, embeds, and connects parallel developments of symbolic representations and geometric representations along with cognitive development theories such as reification, or APOS theory. And finally it systematizes the transitions between various representations of vectors.

Validation of the Framework

For the initial validation of the proposed configuration of vector representations, we used several literatures about multiple representations, semiotic representation, cognitive development theories in mathematics education such as process-object encapsulation, reification, three world of mathematics etc., and mathematical conceptualization. And then using the surveys and follow up interviews, we checked if students can solve problems developed based on each stage and transition jump on the configuration and we figured out the difficulties. The main method here for the validation was both quantitative and qualitative data analysis of the surveys and interviews.

Classifying Approaches and Difficulties

Student difficulties in learning and use of vectors in mathematics can be identified by the configuration of vector representations in the following way.

Approaches

Student approaches can be divided into two big categories. Analytic approach is the approach that students uses column/coordinate forms of vectors frequently and decomposes vectors into detail numbers or variables in order to construct algebraic method of solving problems. Synthetic approach is the approach that focuses more on the geometric structures of vectors. Instead of getting into the detail, students who take this approach would solve questions with bigger pictures and relationships behind holistically.

Difficulties along Epistemological axis

The critical transitions here are transition (A) from physics to mathematics, transition (B) from arithmetic to algebra, and transition (C) from analytic view to synthetic view. The epistemological aspect of transition (A) is related with equivalence relation in the definition of a vector as a directed line segment. Without understanding of mathematical equivalence relation, one cannot understand of the concept of free vectors or mathematical position vectors that are very important for ontological shift from the geometric representation to symbolic representation. Usually this transition happened in high school mathematics. Watson et al. (2003); Poynter (2004) explained the effective way of transition from the embodied world of mathematics to the proceptual world of mathematics that is corresponding to this transition (A) and more transitions like transition (D) on the configuration. But they were not actually able to see what problems were. For example, their resolution or decomposition example is not related with students' understanding of equivalent vector concept from advanced mathematics. Their "action-effect" ways of teaching can mislead the translation by a vector in such a way that translation vectors are very limited to vectors nearby object and its image.

Transition (B) from arithmetic to algebra occurs actually in the earlier mathematics to students. Transition from arithmetic to algebra is one of the main topics in elementary and middle school mathematics. Even this happened earlier, we still can see if college students actually have that variable thinking in this higher level mathematics with different context. In college level, the major transition here will be transition (C) from analytic view to synthetic view, because it has an ontological shift such as reification or process-object encapsulation in it even though it is an epistemological jump. For example, a geometric object can be studied both analytically and synthetically. Switching back and forth of the ways of thinking and investigating a geometric object is epistemological shift. However, when we use vectors to investigate a geometric object, it is easily seen that corresponding transition from analytic representation to synthetic representation of vectors is related with this epistemological shift as well as 'process object encapsulation' that is classified as an ontological shift. To see the difficulty for the major transition, one can give problems of vector additions and subtractions. If one has gone through process object encapsulation or reification, especially in geometric representations, he or she can think vector additions and subtractions in a structural way. He or she will not do calculations by drawing parallelograms, inverse vectors or by measuring the magnitudes. Instead, he or she will see the answer directly from the triangles or polygons and direction of vectors on them.

Difficulties along Ontological axis

There are three transitions here, transition (A) from physics to mathematics, transition (D) from geometric object to symbolic object, and transition (E) from concrete object to abstract object.

Transition (A) from the ontological aspect is about the transition from vectors as representation of physical quantities that have both directions and magnitudes to vectors as mathematical objects that can be represented by directed line segments. Once this transition is done, vectors will not have any physical attachments such as physical directions or units.

Transition (D) is important transition on this axis, because in high school level, with standard basis vectors, this transition is not hard to be achieved. The coordinates for the end points of position vectors will automatically be the coordinate form of vector representations or become column vector forms. However, at undergraduate mathematics level, this transition implies understanding of the difference between vector space and Euclidean coordinate space. The coordinate of the points will not directly correspond to the representations but the component-wise observation on linear combination of basis vectors will give correct coordinate representations of vectors on Euclidean space. This can be the major transition at undergraduate level mathematics, because it has an epistemological shift such as understanding of basis vectors in it even though it is an ontological jump.

Transition (E) is somewhat easier than the rest if it is restricted in symbolic extension, because symbolically it is just appending one more component on the representation. However, strong connection between geometric representations and symbolic representations sometimes will not help students generalize the vectors to higher dimensions such as four or above. And to understand abstract concept of vectors, it is needed to understand the epistemological role of the inner product in vector space in this transition. In 2D or 3D, both geometrical and symbolic representations assume that the magnitudes of two vectors determine the inner product of two vectors. In 4D or above, or more abstractly, the inner product determines the magnitude of a vector. It is a big transition from concrete concepts to abstract concepts of vectors.

Other difficulties on the first quadrant

Other than above difficulties, difficulties that occur on the mixed stages of the configuration are more complex. If one assumes that the stages are just reflections of epistemological shifts, ontological shifts and their simple mixes, then 'column/coordinate form with variables' will be easily achieved by understanding 'arrows with axes but no scales' and 'numerical column/coordinate form'. However it is really doubtful and hard to see due to complexity.

Mini Study on Student Approaches and Difficulties

Research Questions

By proposing the configuration of vector representations, we hypothesize that students difficulties lies on ontological and epistemological jumps on transitions in the configuration of vector representations, and students tend to use particular representations more and confine their understanding and use of vectors in a few approaches rather than have flexibility in understanding and use of various approaches. Unlike the assumptions from various studies that students have more difficulties in symbolic representation than geometric representation, some students have more difficulties in geometric representations vectors than symbolic representations. Hence, the following will be the research questions that we will investigate in the main study:

(1) What student approaches and difficulties can be identified in understanding and using of vectors?
(2) How useful is this framework to describe undergraduate students' understanding and use of vectors?

As a preliminary report, we explore some blurry snapshots of these in this paper.

Method

This study was conducted during the 2010 Fall Semester at a large, Midwest public university. A task based survey questions that ask student background on vectors, student approaches in representations, and difficulties was given. This portion of the study attempts to focus on two aspects of student performance in solving problems with various vector representations. One aspect will be identifying the representation stages in the configuration that students are able to use by checking five important transitions. The other aspect will be identifying student difficulties located in the transitions among representation stages in the configuration. To explore the configuration, survey data on each transition will be analyzed with descriptive statistics. In order to see the usefulness of the proposed configuration of vector representations, the follow up interview was also conducted. The main purpose of the interview was to check and modify survey questions so that they can capture more effectively student difficulties along with transitions on the configuration. After administering the actual survey, students were selected for interviews. The selected students signed up for an one-hour block of time for their interview on the day and time that was most convenient for them. Interviews were held in a neutral location away from the students' classrooms and were audio-recorded for further analysis. Transcribed interviews were coded and analyzed with the constructed framework and semiotic representation theory.

Participants and Protocol

Twenty-nine students from senior level Capstone in Mathematics course for pre-service secondary mathematics education participated in the experiment. Pre-service teachers are required to study enough advanced level of vector representations. (Engineering students or economics students could limit their understanding of abstract vectors, for example.) 6 questions about their background information such as major, high school courses related with vectors, college courses related with vectors, etc. were asked together with 12 vector questions with topics about transitions, forces, robot arm, origin, non-standard basis, rotations, polygons, a very long sum, cube, triangle midpoint, associativity, point/vector. After the survey, four students were selected for the follow up interviews to probe deeper student thinking.

Results and Discussion

Some interesting results were founded. Among them, this paper will introduce two examples briefly.

First, we could identify *physics representation* in student mathematical thinking of vectors. Students tended to use and think physics representations (pseudo-geometric and pseudo-abstract diagram) even though the questions were nothing to do with physics contexts. Students also had difficulties in transition (A) from physics representation to mathematics representation. When they were asked the following questions about mathematical transformation with translation vectors, they could not connect the equivalence conditions of vectors with vectors representing a geometric translation.

> Question 7. Translation: A translation can be represented by a vector $\vec{v}$, $T_{\vec{v}}(P) = P + \vec{v}$ for any point P.
>
> (a) List all vectors that do NOT represent the translation of triangle A to triangle B in the figure.
>
> (b) Circle all vectors that are equivalent to $\vec{a}$.

Figure 2. Physics to Mathematics Transition Question

Student understanding of the relationship between the represented object (geometric translation) and the representation (directed line segment) of it was not strong enough to connect their mathematical knowledge of vector equivalence and translations. They rather saw a directed line segment as a diagram attached to geometric objects that represented directions and magnitudes of the physical motion of the physical objects. In the question 7 (a), $\vec{d}$, $\vec{e}$, $\vec{f}$ are not related with translation. However, some students who were still using physical representation of vectors interpreted that $\vec{g}$ and $\vec{h}$ were not related with this translation as well, because these vectors were away from actual triangles. $\vec{c}$ was also chosen as a vector not related with this translation because its initial point was on the image of the triangle. The following cluster tree diagram from 29 students responses shows this clearer. (See Figure 3.)

In the follow up interview, a student explained what he was thinking when he tried to solve this question 7(a). His answer was $\vec{d}$, , , :

> $\vec{d}$ comes back, it's a translation from B to A, so it's opposite. is a translation of onward, so $\vec{a}$, A going to B would not include , in my mind. is the correct form up 1 over 3 but it's just way up here and there is a, two separate points than A and B and the same with which is the same thing but just sort of from other points… so if you moved it over [nearer to triangles], then yeah it would be…but I guess, in my mind, it's something totally different.

When the interviewer asked about vector $\vec{b}$ with emphasizing that the vector itself did not touch the sides of triangle A, he responded:

> If you shifted this over here to A. $\vec{b}$ is touching the point on it. Very close……

From his response, we can see that this directed line segment (arrow) representation was not understood as a geometric translation that translate all the points on the plane, but rather as a representation of a displacement or physical motion of triangle A. The distance between the geometric object and the directed line segment was the critical condition of decision-making. His idea of translation was described as:

> Translation is just moving it (a triangle) up 1 unit and shifting that triangle over 3 units, Umm, I believe, yeah that's going to be where it's going to go. Yeah so from A to B it would be going up 1 and over 3.

Figure 3. Cluster Tree from Student Responses on Translation Vector

More interesting fact is that most students answered correctly on the second question about vector equivalence. As we can see from the cluster tree diagram below (Figure 4), even though there seems to be some degrees of acceptance as equivalent vectors, $\vec{b}$, , , $\vec{h}$ are clustered as equivalent as $\vec{a}$.

This result tells us that vector representation should be explored by refining the notion of representations and by integrating different theoretical perspectives used to describe cognitive development in mathematics. In particular, this example of student thinking on a vector representation for a geometric translation in Euclidean geometry can be analyzed from the physical embodiment and semiotic representation point of views.

The analysis considers the role of the physical embodiment in the vector representations, and also utilizes the tools of semiotic representations and cognitive development theory with newly developed vector representation framework. For example, if we can refine the notion of register in semiotic representation theory, physics (physical diagram) register as an additional register to well-known registers such as table, graphical, or symbolic registers in vector representations that take into account how students actually think. Actually, the motion of geometric objects in mathematics is quite different from that in physics or realistic context. Freudenthal (1983) described the distinction between physical motion and mathematical motion. Physical motion is something that occurs to an object within space or plane within time, but mathematical motion should be differentiated from physical motion in three ways: from *the limited object* to *the total space (plane)*, from *within* space (plane) to *on* space (plane), from *within time* to *at one blow*. Because this motion of geometric objects cannot be shown fully on paper, students should use representations to describe how an object should be or has been moved. They can invent various ways to represent the motion. Translations move a figure a fixed distance in a given direction. Translation arrows (vectors) represent the distance and direction for moving the figure. However the meaning of the arrows varies from time to time. It is related with student cognitive development and their interpretation of mathematical world (Watson, Spyrou, & Tall, 2003).

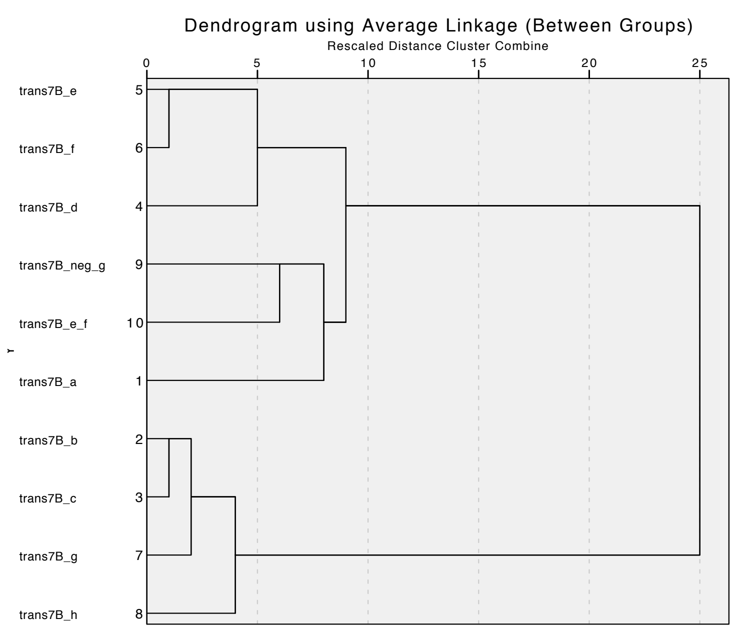

Figure 4. Cluster Tree from Student Responses on Equivalent Vector

The next result we have brought is the *prevalence of analytic approach* in understanding and use of vectors. The following question was related with physics to mathematics, geometric to symbolic, and analytic to synthetic transitions.

Question 16. Triangle Midpoints: In the following △ABC, AB = 2AD and AC = 2AE. We want to show that BC is parallel to DE.

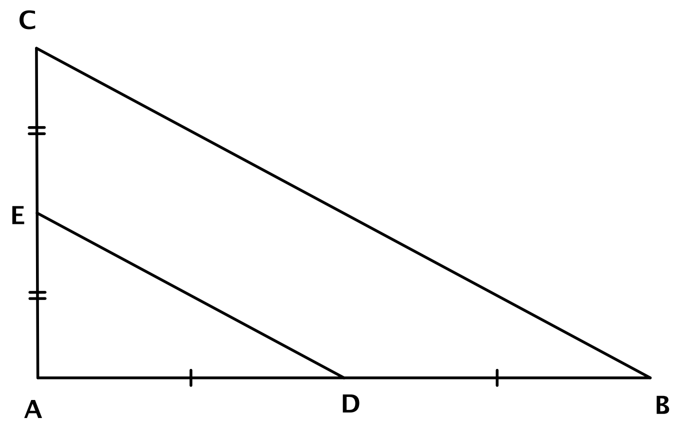

Figure 5. Triangle Midpoints Theorem

Which form of vectors, do you think will be most useful? Circle your answer.

(i) (x, y), (ii) $\begin{pmatrix} a \\ b \end{pmatrix}$, (iii) $\overrightarrow{PQ}$, (iv) $\vec{u}$, (v) Others

(i) and (ii) can be regarded as analytic approach, and (iii), (iv) can be synthetic approach. Because we can put the directed line segment on the sides of triangle and see this question as vector subtraction in the structure sense and scalar multiplication question in the algebraic manipulation sense, synthetic approach is more reasonable. Due to no guarantee on given triangle is right, use of coordinate/column form of vector representation and component-wise calculation through analytic approach is not appropriate in this setting. However, 10 students (34.5%) of 29 students chose (i), and 6 students (21%) chose (ii). 10 students (34.5%) chose (iii), and 0% on (iv). 3 students (10%) gave no responses. Many students attempted this with synthetic approach, but still 55.5% used analytic approach on this question. Their approach was almost about calculating slopes of the sides and tended to put the triangle on a coordinate plane.

As a partial explanation of this, it is believed that U.S. curriculum put more emphasis on the upside down 'L' shape learning path (analytic approach) on the configuration of vector representations whereas European or Asian curriculum put more emphasis on the left handed 'L' shape learning path (synthetic approach) on the configuration of vector representations so that U.S. students would have more difficulties on geometric representations of vectors, even when upside down 'L' shape path on the configuration, can be easily achieved by European or Asian students as well as U.S. students. This requires more comparative research on bigger samples across the countries. From this research, we cannot just say it is because of the curriculum difference without any empirical evidence, and more study about the partial explanation of the difference will be done in the future.

Limitation

The proposed configuration alone as a framework for analysis of student approaches and difficulties only gave us a really blurry idea of what was actually going on in student thinking. It was more helpful to understand the whole range of vector representations that mathematics and mathematics education community required. Holistic and macro view are useful more for learning trajectory or path. In the on-going and future research, we will provide more careful analysis on student thinking using the triple components of semiotic representation, {{representation content, semiotic register used}, represented object} that will focus more on individual thinking and micro view on vector representations. Macro view together with micro view will hopefully coordinate and systemize our better understanding of student approaches and difficulties on vector representations.

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