HOW DO MATHEMATICIANS MAKE SENSE OF DEFINITIONS?

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The purpose of this paper is to share preliminary results from a pilot study on mathematical definitions. Interviews with university mathematicians were designed to gain insight into mathematicians' processes for developing understanding of new definitions. We asked the participants to talk about what helps them understand a new definition and how they support students' understanding of definitions. We also observed them while they engaged in a definition task. Analysis revealed a noticeable difference in the emphasis on examples between what the participants described that they do and what they actually did while working on the definition task. We hypothesize that mathematicians' processes for making sense of a definition necessarily involve considering the definition's usefulness within a particular mathematical setting. Furthermore, these data indicate that mathematicians see examples as a multi-faceted, but not comprehensive, tool for understanding definitions.

Key words: definitions, advanced mathematical thinking, mathematicians, examples

Understanding mathematical definitions is essential for advanced mathematical thinking. However, the evidence suggests that even successful university students with substantial mathematics backgrounds think about and work with definitions in ways that are strikingly different than the practices of mathematicians (cf. Edwards & Ward, 2004; Tall & Vinner, 1981). Research has demonstrated that students prefer working with their own ideas about a concept (i.e. concept image) rather than a formal definition. Concept image is so powerful that it can be difficult to support students in developing an understanding of definitions that is more consistent with the mathematician's perspective. In their chapter on advanced mathematical thinking, Harel, Selden, and Selden (2006) made a case for mathematical definitions research that involves comparing the activity of students with the practice of mathematicians. In this paper we focus our attention on the practices of mathematicians. By doing so, we hope to learn more about what mathematicians actually do when they encounter a new definition, which in turn should inform the ways we support student learning.

Mathematicians encounter definitions in their work in a variety of ways. Definitions are part of the courses mathematicians teach, they are proposed by other mathematicians, and they are sometimes developed as part of the mathematician's own research. In instructional settings, mathematicians must decide how to present definitions to students. In the context of their mathematical work, mathematicians must judge the clarity and appropriateness of stated definitions. When preparing to propose a new definition, mathematicians must also consider presentation, clarity, and usefulness. Each of these settings requires some level of making sense of a given definition within a mathematical setting. We set out to create an interview context in which aspects of this activity were brought out and thus became accessible for analysis.

Specifically, we were interested in gaining insight into the processes mathematicians use to understand a new definition.

We situate our current work within existing literature on *thinking about* definitions while taking a grounded theory approach in our analysis. The ideas we share here contributed to our initial design and overall research question, but we did not attempt to apply an existing framework to these current data. Edwards (1997) drew attention to the lexical nature of mathematical definitions, that the formal statement of a mathematical definition places a concept within a particular class while also distinguishing it from other members of that class (e.g., *continuous function* as a particular type of *function*). Understanding the nature of mathematical definitions is necessary to support advanced mathematical thinking, as can be seen in the practice of mathematicians as summarized by Harel, Selden, and Selden (2006). In particular, their summary includes a list of features of definitions valued by mathematicians. That such a list exists indicates that mathematicians reflect on *definition* as a concept and on its role within the work of mathematics. The practice of mathematicians has been studied in terms of their use of writing, problem solving, and proving (e.g. Weber, 2008). While definitions play a role in each of these activities, the practice of mathematicians centered on encountering a *new* definition has not yet been characterized. Work has been done to investigate students' processes when encountering new definitions (cf. Zaslavsky & Shir, 2005; Zandieh & Rasmussen, 2010). Engaging students in the activities of evaluating statements of definitions and creating definitions seems to increase students' awareness of features of mathematical definitions, perhaps bringing them closer to ways in which mathematicians view definitions. In particular, features that are addressed across students and mathematicians include the need for definitions to be unambiguous, logically equivalent to other definitions of the same concept, hierarchical, stated in a usable form, and should address the purpose for which they were invented (Harel, Selden, & Selden). Definitions make it possible to sort a collection of mathematical objects into two distinct categories, those that satisfy a definition and those that do not satisfy that definition. Activity involving a definition, whether it be evaluating, creating, or using it, necessarily includes exploring representative objects (examples). Michener (1978) outlined distinctions between the roles played by examples within the work of mathematicians. Watson and Mason have written extensively on the role of examples, in particular with respect to student-generated examples (cf. Watson & Mason, 2002), and the potential for developing mathematical thought. We anticipated that asking mathematicians to reflect on their processes of making sense of definitions would involve the use of examples, both for themselves and for their students.

Our work is also grounded in a theoretical perspective on mathematics and understanding (Schoenfeld, 1994; Skemp, 1976). Schoenfeld described *mathematics* as the products of the work of mathematicians, *learning* mathematics as finding out about those products, and *doing* mathematics as creating those products by oneself or with others (1994, p. 55-56). Here we frequently reference understanding and sense making. We see understanding much like it was described by Skemp in that it is comprised of procedural and relational aspects (1976). When students learn mathematics, they are able to replicate certain procedures and recall facts. When students do mathematics, their work becomes consistent with that of mathematicians and results in relational understanding; knowing what to do, why, and how a mathematical idea is related to other mathematical ideas. Sense making refers to the psychological and socio-cultural processes involved in doing mathematics. When mathematicians and students are making sense of a mathematical idea they are developing their understanding of that idea.

Methodology

We conducted interviews with eight mathematicians who were employed at a large university in the northwestern region of the United States. The mathematicians volunteered to participate in the study. Five of the interviewed mathematicians were currently engaged in mathematical research in the areas of applied mathematics, cryptology, geometric topology, and set theory. Three of the mathematicians worked in multiple areas. All of the mathematicians were responsible for teaching at both the undergraduate and graduate levels and had experience that ranged from five to twenty or more years. Two were currently involved in small research studies aimed at improving instruction. In this preliminary report, we share findings associated with interviews of five of these mathematicians (Adam, Sam, Greg, Sadie, and Marc), three of whom were actively involved in mathematical research (Adam, Sam, and Greg).

The interviews were structured to gather data on mathematicians' thoughts and actions. During the first part of the interview we asked these mathematicians, "What helps you understand a new definition?" and "How do you help students understand definitions?" This part of the interview was designed to encourage them to describe what they do. We used follow-up questions to elicit specific instances of what they did to understand a new definition and what they did to help students understand a definition.

During the second part of the interview, we invited the mathematicians to engage in an example-generation activity and a definition task. The example-generation activity was taken from Watson and Mason (2005, p. 22). We asked the mathematicians to work through this task and then comment on its usefulness for teaching and learning definitions. For the definition task, we asked the mathematicians to talk aloud about what they were thinking as they attempted to familiarize themselves with a definition of formal language. (See Figure 1.) They were made aware that additional information was available upon request, which included definitions of related terms, like *support* and *formal power series*, and an example of a formal language. In addition, we explained that the goal during this part of the interview was to understand their processes, not for them to achieve understanding per se.

Figure 1. Definition of formal language

A *formal language* is the support of a formal power series over X^* where X is an alphabet. (Salomaa & Soittola, 1978)

The interview data were analyzed using a grounded theory approach described by Glaser and Strauss (1967). The interviews were transcribed to support analysis. We focused our initial analysis on the first two interview questions ("What helps you make sense of a new definition?" and "How do you help students understand definitions?") and on the definition task. In analyzing responses to the questions, first the researchers individually identified key aspects within a participant's articulations. Then, these individual aspects were brought to the group for discussion and negotiation of emergent categories. For the definition task, we worked in pairs to create a characterization of the participant's work on the task. These characterizations were then shared within the group and common categories were identified. We then began the work of comparing these categories across the mathematicians' articulated processes for themselves, for their students, and what we observed in their work on the definition task.

Results

Our presentation of results follows the organization of the analysis; that is, focusing on the mathematicians' responses to the two interview questions and their work on the definition task. In each section, we share brief excerpts to illustrate the emerging categories. In their articulations of their own processes, mathematicians clearly identified the use of examples as prominent. We were struck by the consistency between the mathematicians' descriptions of their own processes and their descriptions of how they structure their work with students. We initially noticed a lack of consistency between these articulations and our analysis of their work on the definition task. In particular, the use of examples shifted noticeably from a prominent place in their descriptions to a much more subordinate place in their work.

What I say I do. In response to the first interview question, the mathematicians shared what helps them to understand a new definition. The immediate response for four of the five participants included a reference to examples. This is exemplified in a quote from Adam, "I immediately try to think of examples of what would satisfy the requirements of the definition and what not." Adam stated that he has a set of "standard examples," such as the real line or the set of integers under addition, and that he uses these examples as test cases when he encounters new definitions. This testing process usually leads him to wonder "what things don't have these properties" and contributes to his understanding of the key features of the definition. Sam referred to "specific concrete examples," but acknowledged that finding such examples becomes difficult with abstract objects. Greg and Sadie both cited using examples and added that drawing diagrams or pictures is also helpful.

We asked the mathematicians to recall and share, if possible, an instance where they needed to make sense of a new definition. Adam, Sam, and Greg provided instances from their current research work. Examples played a key role in each of these instances. Adam described encountering the definition of a particular type of bounded group. He stated that he checked his standard examples (e.g., the real line) to see if they had the specific property. He also read a related theorem and realized that he could use his simple examples to build complicated examples that would have the desired property. However, this led him to wonder if there were other examples that did not arise from his simple examples. Adam indicated that examining a range of examples was necessary to understand what the definition intended to "capture."

Sam shared an instance in which he and his colleagues encountered collections of sets that intersect in a particular way. They wanted "to be able to find a definable family of objects with the property. So, it is not enough to know that there is one, we want to be able to isolate it somehow." Through looking at many varied examples, he stated that they identified the necessary conditions for the particular idea. Similar to Adam's process of understanding a newly encountered definition, Sam's newly created definition needed to capture the essential features of the collected examples.

Greg's instance came from algebra in relation to his work in topology. He shared that there are "humungous algebraic definitions that involve algebras with certain structures." He explained that since his work is primarily in geometry, his first inclination is to draw pictures. "So, even though those are algebraic definitions, I can still do what I like to do, namely draw my pictures." He described the process of "transform(ing) the definition into something more visual" so that he can "sit down and try to work out from the example what is really the essential feature of the definition." Across these three instances shared by the mathematicians, examples were prominent in their sense-making processes.

While explicit references to examples were evident in the responses, other aspects of the responses provide insights into more general processes. Marc and Sadie referred to seeing the definition "in context" as part of their process of sense-making. Sadie also mentioned connecting the definition to "other words that I might already know." Adam shared that part of his process may involve "looking things up in books." Sam noted a useful question: "Is this some notion that will actually be useful to prove the kind of results you are interested in?" Responses also included the need to attend to notation and language. Adam noted that one should pay particular attention to any quantifiers involved in the stated definition. Marc stated that the "phrasing has to get parsed; torn apart."

From these data, mathematicians' descriptions of their sense-making processes place examples in a prominent role. In particular, examples allow them to determine the essential features of a definition and to explore the boundaries of the defined collection of objects. We also get a sense that their processes involve active and purposeful engagement over a period of time. Each mathematician indicated that developing understanding of a definition requires focused attention on different aspects of the definition. The words, notation, context, and how particular terms are related to what they already know are important considerations for these mathematicians. Examples seem to serve as a primary method for focusing on one or more aspects of a definition. Furthermore, we see that mathematicians do not expect to necessarily have all the answers per se. Consulting texts and other sources for examples and other information was an important part of their articulated processes.

What I say I do for students. In response to the second interview prompt, the mathematicians talked about what they do to help students understand new definitions. We were struck by the similarity between their descriptions of their own processes and what they stated that they do in their work with students. Adam and Marc both explicitly stated that they attempt to match their students' experience with their own. Adam said, "I try to kind of simulate for them this process I usually go through when I try to understand something." The other participants, while not necessarily explicitly stating this intent, described activities that paralleled their descriptions of their own processes. Within these descriptions, participants referred to connecting new ideas to familiar ones and to the use of pictures or diagrams to help with visualization. Adam and Sam discussed the use of questions to focus students' attention on the relevant features of a definition.

Consistent with their responses on their own work, the participants referred to their use of examples to support students' understanding of concepts. Adam described two instances of using examples to introduce definitions in a number theory course. To introduce the idea of *isomorphism* between groups, Adam described a process of beginning with two groups that seemed different but that have "identically the same structure." He described using the examples to focus students' attention on that common structure before "writ(ing) down the precise mathematical criterion for this concept." Once the definition was stated, Adam described engaging his students in applying the statement to an example involving two groups that appear to be similar; his stated reason is that it is harder "to show that things are not." The second instance he described also used examples, but in this case he chose to present the stated definition first. Adam recounted how he presented the definition of a *group* and then led his students in a discussion using familiar examples of sets and operations to "explore which of those very familiar operations are really group operations and which ones are not and why." In both instances, Adam's description focused on the use of examples to highlight essential features of the defined concept.

Sam described a similar process more generally, "…part of giving them lots of examples is to hopefully turn this new thing into something somewhat mundane." Greg discussed the need to choose the right example, not too trivial but not too complicated, in order to focus attention on the essential features. Sadie "on a regular basis" gives examples of where definitions do not apply: "give an example where the limit definition doesn't work so you don't have a limit or where the function is not continuous and what part of the definition goes wrong." Adam summarized his use of examples and non-examples by saying, "I think it's important that one doesn't just do the positive, you know, where you confirm that something has the property. But, you also investigate the negative to see that, well, it's not just that everything is like this." Sam also shared a similar observation. "So, definitions exclude things. So, a definition tells you, okay, this is what we are looking at. But it also should tell you this is what we are not looking at."

Other responses referred to connecting the new definition to words or contexts that participants thought should be familiar to students. Greg described using real-world analogies to motivate the introduction of mathematical concepts. Sadie cited the usefulness of pictures to connect to familiar ideas. Marc explicitly referred to connecting to familiar terms. These references seem to be concerned with building a web of connections between mathematical terms, visual representations, and notations.

In discussing his own process, Sam commented on knowing how a concept might be used as part of building an understanding of the concept, "…because these might be very abstract objects, but then it might be useful to see, in the abstract setting you are working with, how is this used." This notion is paralleled in his articulation of his work with students. He commented that seeing this mathematical purpose can be difficult for students. "It might be they just don't have enough of the mathematical background intellectually to show the usefulness of these notions. … Eventually, if you are lucky in the course you are teaching, you get to some actual situations in which the notion, the new notion is actually useful." To address this, he gives students references to papers or books in which the notions are used. Sam stated that seeing the concept used for a mathematical purpose can support students' understanding of the idea, "Sometimes you are not going to, to gain any understanding of the definition from a specific example, or from a specific diagram, but only by really using it."

The mathematicians' articulations of their work with students clearly paralleled their descriptions of their own processes. Examples occupied a central role in how they described what they did to support students' understanding of definitions. The use of examples was supported by explicit connections to familiar ideas or images and by showing how a new idea might be used.

What I did. For the definition task we presented the mathematicians with a formal definition (Figure 1) from theoretical computer science with which we anticipated they would have limited familiarity. Our goal was to observe, in some sense, their initial sense-making processes. It was made clear in the interview that achieving understanding of this particular definition was not the goal; rather, we were interested in their thinking and in their reflections on their processes. Participants were aware that additional information related to the definition was available and would be provided upon request. This information is provided in the appendix for reference. An example was part of this additional information. However, not all participants knew that an example was available. Our analysis began with trying to capture the nature of the processes for each participant. We then looked across participants for common categories. Two broad categories emerged, (1) engaging in a decoding process and (2) connecting to familiar ideas or contexts. Given the prominence of examples in the first part of the interview, we also looked for references to examples in general or the use of the provided example.

The mathematicians began with a decoding process that involved obtaining additional information in the form of the supporting definitions. The participants differed in their apparent familiarity with terms within the initial definition and within the supporting definitions. Upon reading the initial definition, Adam offered that “this X is probably a set of things.” He was given the definition for X* and was able to confirm “so this is the set of all finite sequences.” Sam’s initial process was similar: “So, I think the first thing I will ask is what the definition of support is.” After reading the definition for support, he stated, “I guess I immediately begin making some guesses as to what the other words mean.” In some cases the decoding process focused on particular notations; Marc’s first reaction to the initial definition statement was to note an unfortunate typesetting difference between the two X’s on the page. While the typesetting was an error on our part, it revealed an aspect of the decoding process, that of understanding the use of notation and what it may imply about the concept involved.

The decoding process seemed to serve the purpose of identifying components that could be connected to familiar ideas or contexts. In some cases, this focused on particular uses of notation and what that might indicate about the intended context. For example, two of the supporting definitions included “ ”, which seemed to lead Marc to question whether this referred to an interval or an ordered pair. In other cases, the supporting information allowed the mathematician to test connections to familiar contexts. Sam stated he was not seeing the use of the familiar *power series* in this context: “Okay, so I guess the one word that is not coming is what it means that a formal power series is over some set. So this is the word that I don’t know how to apply in this context.” For Adam, the supplied definition of *monoid* verified his conjecture, “Associative and an identity, okay. *Interviewer: Is that what you thought?* Yeah, and that one is okay.” While some participants noted connections to familiar ideas and notation, none felt they achieved understanding of the definition. Adam stated, “So, at this point I’m not satisfied that I understand the definition. I will feel more comfortable when I’m at the point where I can actually reproduce these various things.”

Given the mathematicians’ emphasis on examples in the first part of the interview, we were initially surprised that references to examples did not come up earlier in their work on the definition task. Not all participants knew that an example was one of the pieces of available information; nevertheless, none of the participants asked for an example nor immediately introduced the need for one. For Adam and Sam, the example was offered when they seemed to reach a point of needing additional information that was not available:

> Adam: I am still confused about what is the semiring situation. And what is the zero. I can guess the zero is going to be the empty word, but what are the operations here? There’s supposed to be two operations.
>
> Sam: I don’t know when it says that this is a formal power series over X*, then it means that X* is the domain of the power series. Is that what it is saying?

When the available example was provided, Adam and Sam seemed to connect aspects of the example to familiar notions they had already identified, but having the example did not appear to immediately clear up their remaining questions. Adam said he would “try to think of things I would have thought of as a formal language and see if I can fit it into this.” Sam stated, “this is much more general than I thought" and went on to say that he would need to see "what kind of operations one is interested in doing with language" in order to justify the level of generality.

The other participants' reactions to the provided example varied. Greg felt that he had achieved some level of understanding of the components of the definition and "now would have to do an example." In particular, he would need his example to help determine the importance of the semiring, a component of the definition he had not yet clarified. Sadie, after having asked for supporting information, noted that the interviewer still had an additional piece of paper. When presented with the statement of the example, she began a decoding process in a way that was similar to her work on the definition statement. When offered the example, Marc reflected on his processes: "That might be useful. And pretty clearly, my initial tendency was not in that direction. Certainly that's one way to approach a definition, instead of trying to understand its component parts." Adam, Sam, and Marc attempted to match the example with aspects of the definition. At the end of the interview the mathematicians stated that they would need additional information or to see how the definition would be used to feel more comfortable with the ideas.

While conducting the interviews, and in our initial analysis, we were struck by the consistency between what the mathematicians said they did for themselves and what they said they attempted to do for students and the apparent lack of consistency between what they said they do and with their work on the definition task. That is, the use of examples was a clear theme in what they described about their own mathematical work and their work with students. The mathematicians were articulate about the role that examples can play in developing understanding. However, when presented with a unfamiliar definition, the possibility of using an example to make sense of the concept was not part of their initial processes. We propose two themes from the data to explain our observations.

Themes

Mathematicians' processes for making sense of a definition necessarily involve considering the usefulness of the definition within a particular mathematical setting. Placing a definition within a setting involves a progression of previous definitions, notations, and examples. When presented with the unfamiliar definition, participants began by sorting through the specific terms and notations within the statement. This involved requesting and receiving various supporting definitions. Participants also made references to contexts with which they were familiar that contained terms or symbols used within the definition. In particular, participants questioned things such as why some terms were presented in a specific way, or whether the definition needed to be as general as it appeared to be. None of the participants asked to see an example as part of these initial processes. We see this as the mathematicians needing to situate the statement of the definition clearly within a mathematical setting to judge the value or usefulness of the definition.

Mathematicians see examples as a multi-faceted, but not comprehensive, tool for understanding definitions. In discussing both their own work and their work with students, participants spoke about examples as key to understanding. In particular, they noted that examples should be chosen carefully so that they serve to draw attention to important aspects. In making sense of definitions, the participants said they use examples to confirm their understanding, often choosing "messy" examples to be sure they had not introduced any inappropriate assumptions. Creating or considering non-examples was considered an essential component of understanding as articulated by Sam "…the only way to get there is look at concrete examples and look at concrete non-examples." The use of examples and non-examples seemed critical for the mathematicians' own understanding and how they went about supporting their students' understanding.

Our analysis related to this theme is ongoing. Some references to examples are more difficult to analyze than others; it is not always clear if when talking about "examples" the mathematicians referred to representative objects or to 'worked exercises' in the sense of using the new definition to accomplish some task. Nonetheless, we identified three potential facets of how the mathematicians described using examples as a tool for understanding: creating shared experiences, developing precision, and using test cases. Evidence for these facets come from participants' responses to the two interview questions and from their reflections throughout the interview. We share a few brief excerpts to illustrate our current thinking about these facets.

Examples can be used to create shared experiences working with a particular concept. Adam described how he used examples in his number theory class to explore the idea of *isomorphism*: "So we started with these two groups and we first played a little bit with this one, [I] tried to lead them to some of the essential properties." Sam stated a similar idea in reference to his own work: "Definitions come from isolating particular examples so that by the time you reach the definition, you have already acquired, hopefully, certain intuition about what this is." In addition, Greg described using a sequence of secant lines as a visualization to support students' ideas about tangent lines. From this, we infer that mathematicians sometimes use examples to familiarize their students with a particular concept. The intent seems to be to engage students in a process of exploration that is similar to what some of the participants described that they do for themselves. These shared experiences create opportunities for reflection upon the concept and upon aspects of the formal definition.

Examples can be used to develop precision. Participants repeatedly used words and phrases such as "capture," "know you have it," and "essential features" in their conversations with us. Such phrases seem connected to the lexical nature of mathematical definitions; that the formal definition clearly sorts mathematical objects according to precise criteria. The participants' references to using non-examples also seemed to be oriented towards developing precision. Several participants noted that it was not enough to know that an object was not a representative of a defined collection of objects. They expected their students (and themselves in their own work) to know which aspect of the definition was not satisfied.

Adam was explicit about how he uses test cases when he encounters a new definition. In both describing his research work and his instruction, he referred to thinking through a familiar set of objects and determining whether or not each object met the conditions of a given definition. This seemed to be a regular part of his initial process of exploring definitions. He noted that he acquired this "list of standard examples" by "playing with mathematics for years." That is, with experience, he developed both this particular process of exploration with test cases and expanded his repertoire of familiar objects.

In these interviews, mathematicians worked to place an unfamiliar definition within a particular mathematical setting. By way of contrast, the mathematical setting is already set in their general practice and in their instruction. When teaching, they attend to presenting ideas in a logical progression so that students have the necessary pieces to understand definitions as part of a particular mathematical setting. In their own work, they are familiar with current terms and notations within their field, so the setting and relationship between mathematical objects are understood. Within each setting, examples and non-examples serve to spotlight key features of the definition; using a non-example can help to clarify why certain aspects of the definition are needed; e.g. why does the function need to be defined at a point to be continuous at that point?

When we presented mathematicians with an unfamiliar definition, it was not clearly situated within a particular mathematical setting. They needed to understand the key components of the statement to establish the setting before they could use examples as a tool for understanding. That is, examples do not "carry" the definition entirely. Marc summarized this for us, "It's hard to ask for [an example] if you don't know what you're asking for."

Discussion

Our preliminary analysis indicates that mathematicians use a range of processes when making sense of a definition and that these processes are consistent with what they attempt to do in support of students' understanding of mathematical definitions. While the use of examples seems to play a prominent role in this work, examples are used in various ways and do not capture the full range of these processes. We can see, for example, that mathematicians expect understanding of a definition to take time and require focused attention on different aspects of the definition. Mathematicians use examples to explore the boundaries of a definition, establish key features of a definition, and as test cases. They ask questions to focus either their attention or their students' on a key feature or application of a definition. They consult resources when their questions go unanswered. Furthermore, we see the importance of knowing how the definition will be used within a mathematical setting. The participants described how they establish the purpose and context of a definition in their teaching by creating shared experiences working with examples. These examples are chosen to help students gain familiarity with new ideas in relation to what they already know. They also engage students in using definitions. In this way, the mathematicians attempt to replicate the processes for understanding that are an integral part of their own mathematical work.

The participants may not have been aware of the role of purpose when describing their processes of understanding. This is likely due to the fact that their work is usually accomplished within a very well-defined context. Our choice to present a formal definition outside familiar contexts for our participants allowed this aspect to become apparent. The juxtaposition of the mathematicians' articulated process with our observations of their actual process (albeit in an artificial setting) sets the stage for deeper analysis. Within a well-defined context, the role of examples may become more prominent, allowing that to be the focus of one's articulated process. With this observation in mind, we can now return to the mathematicians' descriptions to clarify references to purpose and context. Among the valued features of definitions (Harel, Selden, & Selden, 2006) are that definitions should address the purpose for which they were invented and that definitions should be hierarchical. The participants may not have made explicit references to these features; however, attention to such features may be implicit within their descriptions.

Since our observations are based primarily on what the participants described as their processes that necessarily places limits on any general conclusions we might make. Moreover, the participants' descriptions were somewhat varied and were influenced by what they were ready and willing to share with us. Although the descriptions of their processes were varied, we observed significant themes across the aspects of their work that were common to the participants. Our preliminary analysis of these data provide a broad range of snapshots into mathematicians' sense-making processes related to understanding definitions. The structure of the interview brought an implicit aspect of mathematicians' processes into focus, allowing for a more comprehensive characterization.

Future Work

As a preliminary report, we conclude with future tasks and directions. Analysis of the interviews with the other three participants is currently underway. We have not yet analyzed the interview data on the example generation task. Participants worked through and reflected on the task. These data may provide further insights into the mathematicians' perception of the role of examples, both for themselves and for their students.

Another task ahead of us is to go back to the data and verify (as best we can) exactly what the mathematicians were referencing when they used the word "example." We want to clarify when they might have been thinking of a 'worked exercise' as opposed to representative object. We anticipate using the work of Michener (1978) and Watson and Mason (2002; 2005) to assist us as we analyze these references to examples.

In addition, we conducted a parallel set of interviews with students. Students from a variety of mathematics courses were asked to describe how they make sense of definitions and to provide a description of an instance where they made sense of a definition. We plan a similar grounded-theory approach to the analysis of these data in an effort to reveal common categories that can be used to compare or contrast with those found in the mathematicians' data.

Finally, we note that context became a salient aspect within the interviews with the mathematicians. In some sense, the presented definition may have been too removed from familiar contexts for us to clearly see the participants' sense-making processes. Using a definition more closely related to their mathematical experiences may enhance our understanding of the mathematicians' approaches to making sense of definitions.

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Appendix

Monoid: A *monoid* consists of a set *M*, an associative binary operation * on *M* and an *identity* element 1 such that 1*a = a*1=a for every a.

Semiring: A *semiring* is a set *A* together with two binary operations + and * and two constant elements 0 and 1 such that

1. <*A*,+,0> is a commutative monoid,
2. <*A*,*,1> is a monoid,
3. The distributive laws a*(b+c) =a*b + a*c and (a+b)*c = a*c + b*c hold, and
4. 0*a = a*0 =0 for every a.

Formal power series: Let *M* be a monoid and *A* a semiring. Mappings *r* of *M* into *A* are called *formal power series*. The values of *r* are denoted by where $w \in M$. We denote *r* by

$$r = \sum_{w \in M} (r, w)w \,.$$

Support: $\{w \mid (r,w) \neq 0\}$ is called the *support* of and denoted by *supp(R)*.

Alphabet: An *alphabet X* is a finite nonempty set.

$X^* = \bigcup_{n=0}^{\infty} X^n$ where $X^0 = \{\lambda\}$, $X^2 = \{xy \mid x, y \in X\}$, and for n>1. Note λ is called the *empty word*.

Example: Let $X = \{a, b\}$, then $L = \{a^n b^n \mid n \in N\}$ is the support of .