A Multi-Strand Model for Student Comprehension of the Limit Concept

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Students demonstrate a variety of behaviors while solving problems involving limits. However, no model currently exists to address what students do understand about limits while accounting for these varied behaviors. This study presents one possible model. This paper proposes that student behavior while solving limit problems may be interpreted by strands that reflect the student's method for solving a problem involving limits, the student's justification for the solution, and the applicability of the student's method and justification within the context of the problem. This paper concludes with the presentation of a model which demonstrates how these strands connect to each other and a proposal of future directions for its refinement.

Key words: limits, student understanding

Introduction

Much research has focused on the misconceptions held by students as they navigate the concept of limits in their first year calculus courses (Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Szydlik, 2000; Tall, 1992; Williams, 2001). However, very few studies seem to identify what students actually do appear to know about limits and the ways in which they demonstrate this knowledge. Hence there is a need for a model that would allow us to determine what students actually know and understand about limits.

Exacerbating the situation further is the condition that, depending on the question, students may exhibit a range of behaviors they make sense of the question and develop a solution strategy (Koedinger & Nathan, 2004; Schoenfeld, 1985). Any attempt to create a model of what students know about limits needs to take this issue into consideration. Despite the inherit difficulty of accommodating the different responses that may be elicited from a student by two differently phrased questions, there are several lenses available in the current literature that give a starting point to the construction of such a model.

One lens addresses the method that a student chooses to utilize in solving a problem involving limits. The Harvard Calculus Consortium's "Rule of 4" (Knill, 2009) aligns with the four main modalities that students exhibit as they work on limit problems.

Another lens looks at the students' justifications of their methods for solution and their subsequent work to produce the solution. Alcock and Weber's (2008) discussion of syntactic and semantic proof strategies lends itself to describing the types of explanations produced by students regarding their solutions.

The combination of these lenses along with the identification of their fit within the context of the problem, or their applicability, provides the basis for a multi-strand model to describe student understanding of limits. This paper provides evidence of such a model's usefulness in identifying student knowledge of limits and concludes with a presentation of the model in its entirety.

Literature Review

Out of the tools available for measuring student understanding of limits, the one cited most often is the 7 Step Genetic Decomposition developed by Cottrill, et. al. (1996). It arose as an aid for observing the progression of levels of student understanding of the formal definition of limit in the realistic mathematics education (RME) environment. The seven stages track the student's grasp of necessary processes and his subsequent coordination of these processes with each other in order to develop a full picture of the formal definition of limit. This model was later expanded by Swinyard (2009) by the addition of more detail to stages 5 – 7 of the original genetic decomposition.

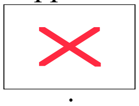
The genetic decomposition was not designed to be a metric or rubric for assessing what knowledge students have about the limit concept. There was no reason to suspect that it would work outside an RME setting, let alone inform us of what a student is thinking. However, it was not unreasonable to believe that there was enough rich description given in the stages to compare to a student's verbalized thought process and possibly assess what the student understood. That is, if students used the language of the formal definition informally as they approached solving problems about limits that did not require the formal definition.

However, the genetic decomposition was not designed to account for the different behaviors that students exhibit when they approach the solution of limit problems that do not involve the formal definition. Hence we need a way to identify and categorize the different student solution methods that arise. One way to do this is to group the exhibited behaviors by their intent through observation of the student's words, symbols, and actions. That is, we can consider the words, symbols, and actions to be external signs of the student's understanding. Thus, in the language of semiotics we can group external signs that indicate similar intent or seem to be the same method into modalities (Saussure, 1966).

The Harvard Calculus Consortium's (HCC) "rule of four" provides a convenient preliminary categorization scheme for these modalities. The HCC was one of the first groups to recognize the importance of being able to approach a problem multiple perspectives and advocate for the inclusion of these perspectives in the classroom. Originally the HCC promoted the "rule of three" and suggested that calculus concepts, such as limits, should be presented graphically, algebraically, and numerically (Knill, 2009). This was later extended to the "rule of four" to include a verbal explanation of the material as well.

If these are the rules that guide explanations of examples in typical calculus textbooks and lectures, it seems reasonable to expect some of these modalities of thought to appear while students approach the solution of problems involving limits. Thus for the creation of the proposed model, the "rule of four" has been adapted from a teaching suggestion into the categories for the method modalities students may exhibit while solving problems.

While the method a student uses to approach a problem may be easy to determine, the explanation he then offers of how he has arrived at the answer can be difficult to decipher. Hence, the model must find someway to interpret the student's justification. One way in which we can look at the student's explanation is to see how it relates to the context of the given problem. Either the student proceeded in a straightforward manner suggested by the context of the problem or the student tapped into knowledge of ideas related to the problem to ascertain an answer prior to returning to the context of the problem. This is the idea developed by Alcock and Weber (2008) when they proposed the idea of semantic versus syntactic proof production.

The syntactic refers to the student staying within the symbolic context given by the problem. Suppose a student was given the problem: "Evaluate the following limit algebraically:
." If the student then chose to manipulate the symbols by factoring the numerator, crossing out like terms, and then equating the statements in order to evaluate the limit, he would be exhibiting syntactic reasoning.

If the student had instead graphed the function to visually ascertain what the limit is doing or if the student had created a table of values to look for a trend in the function's values, he would be using semantic reasoning. Thus, for the purposes of the model, the semantic is when the student moves beyond the symbolic context of the problem to reason about the solution prior to restating the answer in the necessary context.

Method

Using a grounded theory design (Glaser & Strauss, 1967; Strauss & Corbin, 1990), data was collected in two rounds with analysis of the first round of data informing the second round of data collection. Both rounds of data were collected during fall semesters from large lecture, first year, calculus courses. Students enrolled in these courses were typically physics majors, math majors, or engineering majors.

Data Collection - Round 1

A 20 item multiple choice assessment tool was distributed to 55 students. Out of these 20 items, 3 items were determined to be most relevant to the issue at hand. Four students were selected, representing a range of student understanding, to participate in task-based interviews based on their answers to these items.

Task 1.

In this question you are required to encircle the number in front of your choice(s). Which statement(s) in 1.1 to 1.7 below must be true if f is a function for which $\lim_{x\to 2} f(x) = 3$? Encircle 1.8 if you think that none of them is true.

1.1. f is continuous at the point .

1.2. is defined at .

1.3. $f(2) = 3$.

1.4. .

1.5. exists.

1.6. For every positive integer n, there is a real number such that if $0 < |x-2| < \delta$, then $|f(x) - 3| < \frac{1}{n}$.

1.7. For every real number , there is a real number such that if $|f(x) - 3| < \varepsilon$, then $0 < |x-2| < \delta$.

1.8. None of the above-mentioned statements.

Figure 1. Task 1 for the first round of data collection, adapted from Bezuidenhout (2001).

Each interview was approximately 60 minutes in length and followed the same format. Encouraging the use of the talk-aloud protocol, students were presented with two tasks adapted from Bezuidenhout (2001), the first of which is presented in Figure 1, which got at the students' conceptual understandings of limits. The purpose of the tasks was to use conceptual limit problems as a means to explore how well the genetic decomposition described the student's understanding. Of the four students interviewed, one student was unable to complete the tasks. Thus, the results of her interview were excluded from analysis. Of the remaining three interviewed students, two of them were interviewed together. As their insights often played off each other, there may be times when Arlene and Amos[1] are discussed in tandem.

Interviews were transcribed and then coded for evidence of the student's standing within the genetic decomposition. The analysis of this data informed the creation of a 6 item multiple choice assessment tool and a new set of interview tasks for the second round of data collection.

Data Collection - Round 2

During the second round, a 6 item multiple choice assessment, augmented with an "explain your answer" section, was administered to 137 students. Based on student responses to this assessment, 6 students, again representing a range of student understanding, were selected to participate in task-based interviews. Each interview was approximately 60 minutes in length.

The results of the first round of interviews demonstrated that students do draw on a number of modalities for approaching limit problems. Thus the 6 tasks administered during the interviews in this round of data collection were designed to draw out as many of those modalities as possible. The goal was to elicit a wide range of responses from the students by making them think about limits from several different perspectives. These tasks required students to think about everything from "what does it mean for the limit to exist" up through "solve this (unfamiliar) limit problem." Examples of three of the tasks from this round of data collection are given in Figure 2. Students were encouraged to complete these tasks to the best of their abilities using the talk aloud protocol. Interviews were subsequently transcribed and coded for evidence of the student's modality of thinking and the type of justification used.

Task 1: Can you give some examples of a function where $\lim_{x \to -2} f(x)$ exists? Can you give some examples of a function where the limit does not exist?
Task 2: Can you explain why these are (relevant) examples (or non-examples)?
Task 6: (a) How would you evaluate the following limit: ?

Figure 2. Three of the six tasks given to students during the second round of data collection.

Data and Analysis

Although different tasks were used during the different rounds of data collection, both rounds of data offer insight into how students demonstrate their understanding of limits. The students' talk and writing while they worked indicated they were considering a variety of approaches to the problems and often led to more than one attempt at a solution using different modalities. To demonstrate the facility of this model, several students' responses are presented along with the analysis the proposed model allows us to make. The first two students presented come from the first round of data collection and while the remaining two students were participants during the second round.

Amos

Amos primarily exhibited a symbolic modality during his attempts to respond to Task 1. He chose to use the function $f(x) = x$ as an example to help him as he reasoned through which of the statements were true. This led him to some confusion at first. He observed that "So they're saying that as x approaches 2, it's supposed to be equal to 3, but if you replace x with 2, it does not equal 3. So I don't feel that the function's continuous or defined."

Amos's modality was applicable to the problem. The creation of algebraically expressed functions to serve as examples and counterexamples to reason about a given problem shows that Amos understood that it is often easier to discuss limits with a function in hand. His justification during this portion can be classified as semantic as his explanation of his work is centered on his selected example rather than addressing the generalized context presented in the task.

However, his selection of a continuous function that did not fit the context of the problem is a failure of the applicability of his justification. This coupled with his lack of reasoning in the generalized context suggests that Amos may not have a strong conceptual understanding of limits.

Arlene

Arlene started her solution of Task 1 using the verbal modality. That is, as she tried to reason about the veracity of the statements, she spoke her rationalization aloud and left no symbolic record of her thoughts. She said:

> [1.1] should be true because, unless you specify if it's like a right hand or a left hand limit, if there is a limit, I'm pretty sure that the function should be continuous. Otherwise it would have to be like, specified, like an exponential or natural log function or something like that. But I don't think that the second statement, it has to be defined at x equals two, I don't think that one is necessarily has to be. No, that would make the first one wrong.

Recognizing her conflicting statements, Arlene switched to a visual modality to resolve her issues. The graphs she created are presented in Figure 3.

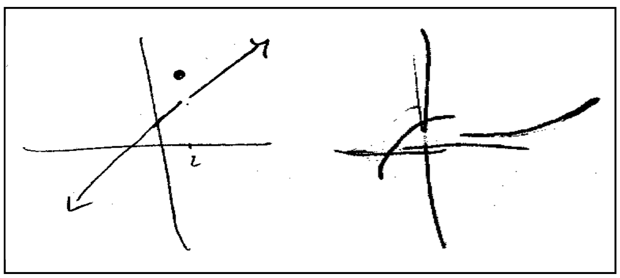


Figure 3. The graphs Arlene drew to reason through Task 1 during Round 1 of data collection.

Arlene explained her new strategy by stating "I was just drawing a graph and I put a dot on it that was supposed to signify three on my little line. I was just thinking, to try and come up with, like an *f* of *x* where it would [...] work for this problem. Like, as *x* approaches two, it would equal to three, just to help me think of some of them."

In Arlene's justifications of her modalities, she stays in the context of the problem by remaining general about the functions she is representing graphically. Although the first graph she drew was a line, her later graph shows that she was also thinking of more generic functions and how they can behave around the value . Thus, her justification is syntactic.

Ultimately, Arlene recognized that both her method and justification were applicable to the problem and used them to rule out statement 1.1. The successful conjunction of all three strands taken together suggests that Arlene has a good understanding of limits in this context. This understanding is further demonstrated through her interactions with her interview partner Amos.

Arlene recognized Amos's selected function as incompatible with the problem. She tried to correct Amos by entering a symbolic modality and utilizing a different equation as she told him to "look at as like any polynomial that could fit this like, I came up with $x+1$." Amos remained unconvinced so Arlene switched to a numeric modality while presenting her graphical examples as more evidence in her favor. She said, "It just doesn't necessarily mean it has to be three at two, because two could be removed. So at like 2.0001 [the function would] still be on that path to approach three."

Barry

Barry also exhibited flexibility in his selection of modalities. When asked to work on Task 1 and 2, Barry started with a verbal description[2]:

> It could have any function that is continuous at x equal to -2. It doesn't have to be continuous at that point, but both lines like this have to at least appear to be going towards the same point even when magnified to some extent. The second situation is a little harder to describe but I think it's the same as just checking whether the limit from the left and the limit from the right are equal.

With very little prompting from the interviewer, Barry was also able to produce graphical examples and algebraic expressions for the functions depicted in the graphs he drew. These examples are presented in Figure 4.

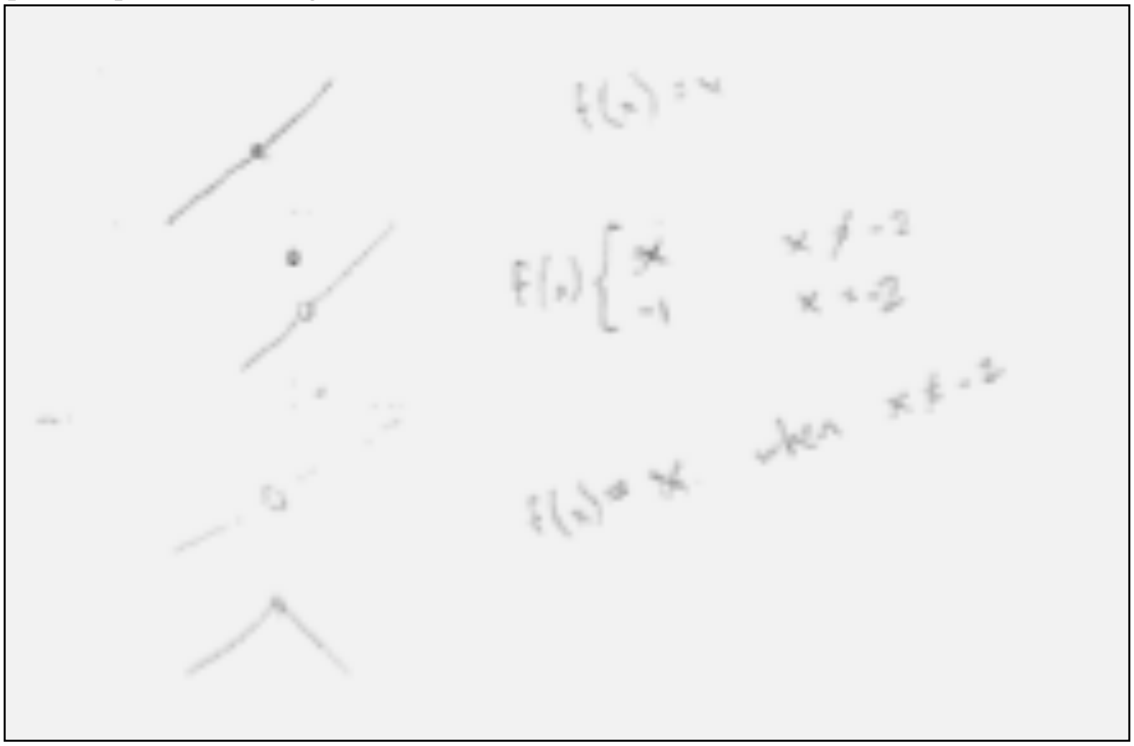


Figure 4. Barry's graphical examples and accompanying algebraic expressions.

While explaining his thought process in the verbal modality and the visual modality, Barry's justification was syntactic. His examples were kept general and represented the families of functions for each case, which was what the context of the task required. When Barry switched to the symbolic modality to create algebraic expressions for the functions depicted by the graph, his justification became semantic with respect to the original problem, as he was no longer being general. However, Barry's justification could still be considered syntactic when taken in the context of the challenge issued by the interviewer, which was to come up with expressions for the functions he drew.

Barry wasn't given much of an opportunity during this task to express whether he recognized the applicability of his justifications beyond the fact that he saw fit to present them in the first place. Thus, Barry exhibited flexibility in selecting applicable modalities to approach solving the problem and followed each modality with an applicable justification. We may be able to conclude that Barry has a strong understanding of the limit concept.

Bud

As he began to respond to the first task Bud also gave a thorough verbal description of when the limit would and would not exist. For instance, Bud said that:

> As long as the point 2 is approached on your function, from both sides of course, then the limit would, as *x* approached 2, would exist […] so basically you can have any sort of shape of line or function or graph or whatever and as long as it approaches 2 from both sides, it'll exist.

Despite prompting by the interviewer, Bud did not feel compelled to switch modalities to re-express his answer. Bud's adherence to the verbal modality in this task may indicate that he was unable to reason about this conceptual question in any of the other modalities. This could be a sign that his conception of limit may have flaws.

During his solution attempts to tasks later in the interview, Bud did demonstrate additional modalities. Bud started to determine the limit given in Task 6a using a symbolic modality. After determining that the limit would yield an indeterminate form if directly evaluated at the given value, Bud opted to apply l'Hopital's rule. Given that within the classroom norms, the implicit context of this problem would be to apply algebraic methods to its solution, Bud's ensuing justification can be classified as syntactic.

When asked how he would verify his answer, Bud asked if he was allowed access to a graphing calculator. After ascertaining that this would have been a viable option all along, Bud declared, "Alright, well then I would have just started out by graphing [the function]," and proceeded to generate the graph of the function. While waiting for the graphing calculator to finish graphing the function, Bud appeared to be struck by a thought, "So we want the limit as *x* approaches negative 3, so [the graphing window] should be looking at negative 3. So it'd actually be easier for me to go to my table."

Thus for this task, Bud switched from the symbolic modality to the visual modality and subsequently to the numeric modality. Each modality was applicable to the stated problem. Additionally, his justifications were syntactic with respect to the phrasing of the task and were also applicable. However, since Bud exhibited distinct modalities that differed with respect to the different tasks he may not have established connections between the aspects of limits depicted in the two different problems. This could represent a flaw in how Bud understands the limit concept.

Results

The data collected from both rounds of the study suggest that there are three interconnected strands enable us to see student understanding as perceived through their attempts to solve problems involving limits. These strands are method, justification, and applicability.

Method and Modalities

The interviewed students seemed to fall into one of four main modalities as they answered questions involving limits: (1) Symbolic, (2) Visual, (3) Verbal, and (4) Numeric. Each of these modalities encompasses certain unique behaviors, recorded symbols, and words used by students that allow us to differentiate between them.

The symbolic modality is denoted by student creation of examples featuring specific algebraic representations of functions, such as those offered by Amos as he worked through the veracity of a list of statements or those created by Barry when he was prompted to identify which functions he had graphed. This modality also accounts for student use of specific procedures or rules learned about in class. One example of this is Bud's utilization of l'Hopital's rule when trying to evaluate the limit presented in Task 6a in Figure 2.

The visual modality is seen in student use of visual aids, predominantly graphs, for evaluating limits or thinking about families of functions for which a particular limit exists. Hence, Arlene and Barry's use of generic graphs and Bud's graphical analysis of the limit problem in Task 6a are categorized as instances of the visual modality.

The verbal modality is observed when students talk through a solution without making a symbolic record. The data collected during this study relied primarily on audio recordings of interviews, so we are unable to further classify the verbal modalities exhibited by Arlene, Barry, and Bud at this time. It is speculated that students utilizing the verbal modality could arrive at their answer through the use of an analogy or metaphor, by simply talking through a procedure that would have been identified as visual, symbolic, or numeric had it been recorded on paper, or by incorporating gesticulations or other instantiations of perceptuomotor behaviors. A change in data collection strategies will be necessary to record the subcategory of percepuomotor behaviors.

Finally, the numeric modality is evidenced by the student's preference to create tables of values to examine for trends or to more informally plug in several numbers close to the value of interest in order to develop an approximate answer. Though this modality rarely arose as a first choice modality, it often served the purpose of verifying an answer. This is seen in Arlene's use of values close to 2 when trying to convince Amos of the applicability of her graph and also in Bud's approach to answering Task 6a. One reason for the numeric modality's status of second choice may be that the students feel it is not an approach that would be accepted on a homework or exam.

All of these modalities depict the different perspectives a student can hold about limits. Thus each modality has information to contribute regarding how a student has made sense of the limit concept and the approaches that are useful or worthwhile when solving problems involving limits.

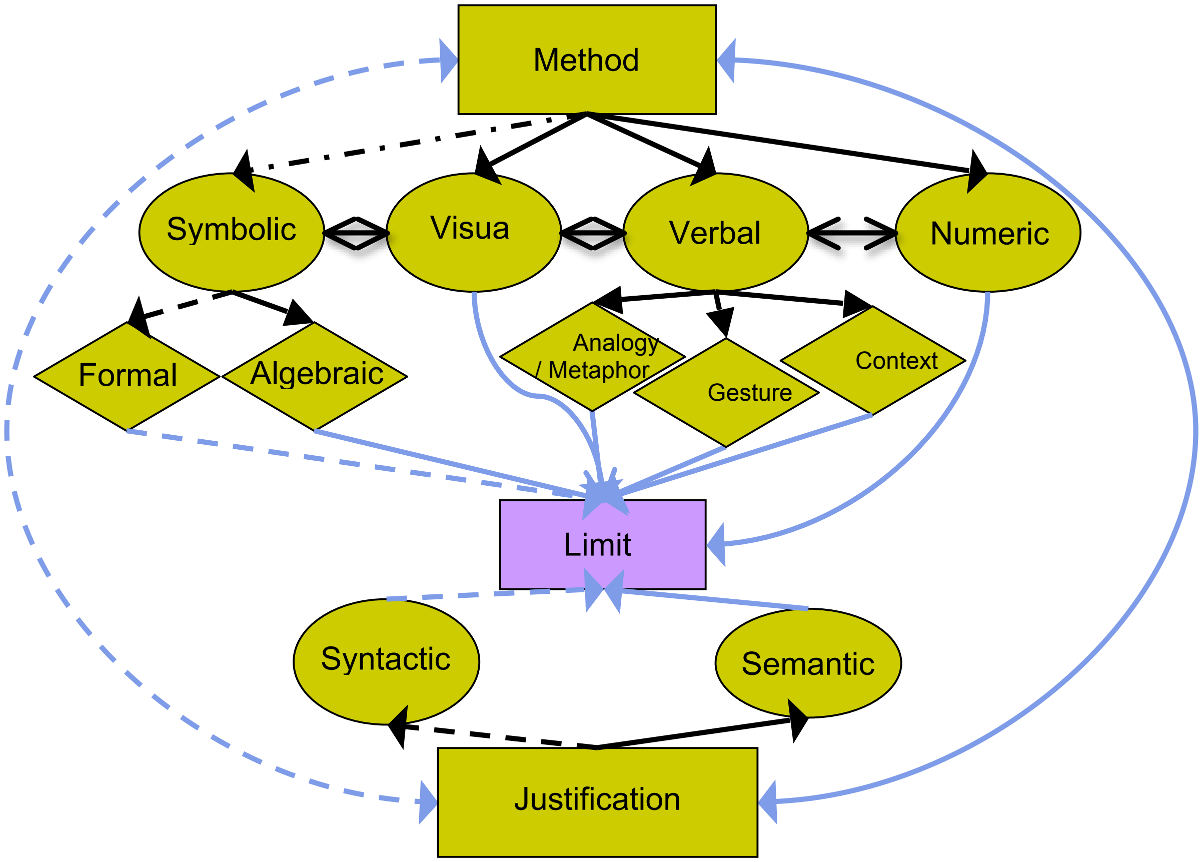
Justification

The method alone is not enough to demonstrate student understanding. Just as important as the student's selection of method is his ability to adequately explain both his work and his selection of method. In particular, justification seems to break down into two main components: whether the student stayed within the framework set up by the question (syntactic) or whether the student translated into another representation to answer the question (semantic).

The students that addressed Task 6a by applying procedures such as multiplying by a conjugate or utilizing l'Hopital's Rule were operating within the implicit framework of the question. Although the phrasing of the task did not explicitly state that an algebraic method must be used, similar limit evaluation questions given on homework and on exams often require such answers. Thus the implicit context was created by the students for themselves. From the students' point of view, the problem was given as an algebraic expression so their work remained algebraic in nature which would be labeled as syntactic justification.

When Bud chose to graph the function and look at a table of values, he left the self-imposed implicit context of the problem and translated into another representation in order to determine the answer to the question. Bud's additional justification during this task could therefore be considered semantic.

Justification also addresses the correctness of the explanation. There is less value to be had in a student that can apply all the algebraic manipulations he was taught but does not realize why the algebraic manipulations allow him to arrive at the correct answer. In a sense, the correctness is the applicability of the justification to the limit at hand.



Applicability

Applicability is the glue that binds a student's method and justification both to each other and to the concept of limit. This strand is important. It is what kept Amos from making progress on his own justifications until Arlene pointed out the error in his thinking. Applicability also arose as students in the second round of data collection debated whether l'Hopital's rule could be used to determine certain limits.

Thus applicability addresses several important issues such as whether the selected method can lead to a solution of the problem or can accurately describe the limit in question, whether the justification validates the method, and whether the student recognizes that his method and justification applies in the context of the problem. If the link of applicability fails to be made along any of these connections, it could indicate that there is a gap in the student's understanding. Then from the instructor's perspective, a failure in applicability should act as an indicator to discuss the solution with the student to try to ascertain where the gap in understanding lies. Even more powerful would be whether the student is able to identify the inapplicability of a method or justification. In this case the gap in knowledge could be more easily accessed. Unfortunately, applicability from the student's perspective can be very difficult to gather data on.

5.4. A Model

It is clear that these three strands are interconnected and relate to each other in an important way. One possible visual representation of the model is presented (below) in Figure 5.

Figure 5. A model of the interconnected strands that comprise student understanding of the limit concept.

In this model, the strands of justification and method are given their own boxes to better visualize their relation to their subcomponents. Applicability then becomes the strand, depicted in blue, which connects the major components of the method strand, the justification strand, and the limit concept. As identified earlier, a lack of connection during a student solution could indicate a flaw in the student's understanding.

The black lines indicate the pieces that are subcomponents of the primary strands of method and justification. Hence justification is either classified as semantic or syntactic. The method strand is then split into the four modalities identified earlier. The further decomposition of the symbolic and verbal modalities merits a brief explanation.

Evidence has been gathered suggesting that the symbolic modality encompasses algebraic behavior. Based on the literature regarding the genetic decomposition (Cottrill, et. al., 1996; Swinyard, 2009) it seems reasonable to stipulate student use of the formal definition could arise given a problem with the correct context. By the very nature of the formal definition, it is most likely to occur in conjunction with the creation of a symbolic record which would necessarily make it a subcomponent of the symbolic modality.

Similarly, due to the limitations of the data collection of this study, the recorded evidence only suggested the existence of the subcomponents of contextual and metaphoric behavior within the verbal modality. Additional data is needed to record physical perceptuomotor behavior exhibited by the student. As such actions would most likely not leave a symbolic record this behavior would fall within the verbal modality.

The dashed lines indicate the connections that are most likely to come into play if a student were to reason through a problem using the formal definition of limit. It falls under the syntactic mode of justification as the language of the formal definition rarely comes into play unless the context of the problem specifically requires it.

Discussion and Implications for Future Research

It seems that different tasks elicit different method modalities for students. If a preference for one modality or a lack of proficiency with a modality were detected, it could show gaps in the student's understanding of the limit concept. At this point it remains a reasonable conjecture that a student that demonstrates greater facility and flexibility when switching between modalities may in fact possess a stronger understanding of the limit concept than a student that is stuck on one modality. Thus the method students select to approach answering a question involving limits needs to be taken into account as it provides one avenue for looking at how the students think about limits in the context of certain problems.

While many first year calculus courses strive for understanding of limits to incorporate the formal definition, it seems students can be successful at determining limits without being able to articulate the formal definition. Hence when exams are designed or we look to formative assessment to inform how we set up our next lecture, we should take the different

modalities students exhibit into account. This may require us to include questions that require the students to not only find the answer, but also to verify their work using another method.

The use of the proposed model to observe student solutions and behaviors while solving problems involving limits offers one way in which to view student understanding. However, it also exposes several other modalities that may need to be looked into. In particular, the modality of perceptuomotor, encompassing gestures under the verbal modality, merits additional investigation to determine what it can tell us about student understanding of limits.

The justification strand also deserves additional exploration and attention. It should be determined whether it can be further disentangled from the method strand. Furthermore, at this time only the subcomponents of semantic and syntactic have been identified. Future research should focus on whether these components are sufficient or if there is more that is encompassed by the justification strand that is not adequately addressed by this model.

Endnotes

1. The names of the students interviewed during the first round of data collection have been changed from the short paper (Galle, 2011). The changes are as follows: Elsa is now Arlene, Jim is now Amos, and Opal is now Agnes.

2. Barry's reference to -2 is not a mistake on his part. The interviewer had incorrectly presented Task 1 as being the limit as *x* approached -2, rather than 2, during this interview.

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