USING TOULMIN ANALYSIS TO LINK AN INSTRUCTOR'S PROOF-PRESENTATION AND STUDENT'S SUBSEQUENT PROOF-WRITING PRACTICES.

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This paper provides a method for analyzing undergraduate teaching of proof-based courses based on Toulmin's model of argumentation. The paper presents a case-study of one instructor's presentation of proofs and shows that she was inconsistent in the amount of detail that she included in her written proofs. The paper then describes how that analysis can be used as a predictor of subsequent student proof-writing performance. Second, student work is analyzed using Toulmin's (1969) model for argumentation. The data shows that the students had adopted the modeled proof-writing behavior of the instructor in terms of the type and quantity of elements of argumentation. The method of analysis was developed via research in a lecture-based abstract algebra class, but has applications to any lecture-based proof-intensive course. This method provides one way to link classroom teaching activities to student performance that forces instructors to assume more responsibility for their students' demonstrated end-of-course performance.

Keywords: Toulmin's model; abstract algebra; proof-presentation; undergraduate mathematics teaching

Introduction and Research Questions

There are many critiques of lecture-based mathematics instruction, especially in the context of proof-based courses, which argue that lecture-based teaching is intimidating and that it misleads students about the nature of mathematics (Thurston, 1994; Cuoco, 2001). Others contend that it hides much of the process used in mathematical thinking and makes it difficult for students to learn or develop an appreciation for the discipline (Dreyfus, 1991). Some maintain that it ignores the important role that mathematicians ascribe to ideas such as elegance, intuition, and cooperation (Burton, 1998; Dreyfus, 1999; Fischbein, 1987). It has also been claimed lecture-based instruction is that it is not an effective way to promote student learning of the mathematics content (Leron & Dubinsky, 1995). In fact, there is evidence that undergraduate mathematics teaching is one of most important reasons that students change majors away from STEM fields (Seymour & Hewitt, 2004).

While each of these assertions may be true, "very little empirical research has yet described and analyzed the practices of teachers of mathematics" (Speer et al., 2010, p. 99) at the undergraduate level despite repeated suggestions for the study of university mathematics teaching (Harel & Sowder, 2007; Harel & Fuller, 2009. For example, Mejia-Ramos and Inglis (2009) also conducted a literature search and found only 102 papers that were educational research studying undergraduate students' experiences writing, reading and understanding proof. Of those 102 papers, they found no papers describing how students understand instructor's presentation of proof. That is, "researchers' questions, methods, and analyses have not generally targeted what teachers say, do, and think about collegiate classrooms in an extensive or detailed way" (Speer et al., 2010, p. 105). In particular this paper will:

1) Use Toulmin's (1969) model of argumentation to analyze the proof-presentation of an instructor of undergraduate abstract algebra course.

2) Propose that students in the class will adopt argumentation methods that are modeled by their instructor.

3) Analyze student work to determine whether the Toulmin analysis of teaching does, in fact, predict students' proof-writing behavior.

Studying Teaching

1.1 Studying undergraduate mathematics instruction

There has been little effort to study what actually is happening in college mathematics classrooms, especially lecture-based classrooms (Speer, Smith, & Horvath, 2010). This gap remains despite numerous suggestions for such study (Harel & Sowder, 2007; Harel & Fuller, 2009).

As Mejia-Ramos and Inglis (2009) documented there are few studies that have analyzed the teaching of undergraduate proof-based courses. Weber (2004) described one instructor's teaching of real analysis and classified his presentation of proofs into three categories. Alcock (2009) interviewed five mathematicians about their teaching of an introduction to proofs class and described the types of thinking that they attempt to inculcate in their students along with some suggestions of how they might attempt it. Weber (2010) has similarly interviewed mathematicians about the reasons that they might present proofs and how that relates to their teaching of an introduction to proofs class. Yet, "none of the articles discussed tasks directly focussed on the presentation of an argument to demonstrate students' understanding of it" (Mejia-Ramos & Inglis, 2009, p. 91). In particular, Mejia-Ramos and Inglis argue that, "we, mathematics educators, know very little about students' behaviour in some of the main types of activities involved in the assessment of their proving skills" (2009, p. 93) and argue that research should focus on analyzing the presentation of proofs and how students then understand them.

The research that has been done on advanced mathematics courses has shown that lecture-based instructors consciously model important mathematical behaviors and ways of thinking as part of their teaching (Weber, 2004; Alcock, 2009; Fukawa-Connelly, 2010). I posit that students appropriate some of the modeled behavior, but not always those aspects that faculty believe are most important. This argument is supported by research on transfer-in-pieces (Wagner, 2006). This research claims that novices may not perceive the same aspects of a demonstration or problem as important that an expert would, and instead attend to other, less important aspects. Wagner (2006) demonstrated that students subsequently transferred aspects of a structure from one problem to another that were at times inappropriate.

In a proof-based course like abstract algebra, the instructors model proof-writing strategies and the types of arguments that they expect from their students. We can analyze these instructors' proof-presentation to understand what they expect of students and look to see if students do, in fact, display similar proof-writing habits. It is necessary to keep in mind that the students may have appropriated aspects that were different than what the instructor intended.

After analyzing classroom teaching, this paper turns to student work. The paper provides a preliminary means of linking the analyses of classroom instruction with student learning. To analyze the way the teacher modeled proof-writing, this paper draws upon Toulmin's model of argumentation (1969) as an analytical tool. In particular, this paper focuses on the relationship between the instructor's modeled presentation of data, warrants, backing, and qualifiers and examines how that presentation is reflected in student work. I show that the students' written proofs are similar to those presented by the teacher. Admittedly, this is an imperfect measure, as the students also receive written feedback on homework and examinations, but it is reasonable to assume that the behavior careful instructors model in the classroom is consistent with their written commentary.

This analysis also suggests that analyzing the instructor's classroom presentation of proof is a valid way to explain student work. It reinforces the often-heard refrain, that "students learn what you teach them." Finally, while this analysis is presented in the context of an algebra course I argue that it is applicable to any proof-based mathematics course and especially useful in analyzing links between lecture-based teaching and student proof-writing.

1.2 What do I mean by lecture?

Let me briefly specify what I mean by traditional, lecture-based courses. Rasmussen and Marrongelle (2006) described a scale of teaching that ranges along a continuum from "pure telling" to "pure investigation." Similarly, McClain and Cobb (2001) characterized the spectrum of teaching as running from "non-interventionist" to "total responsibility." The instructor for the class described in this study used a style closer to "pure telling" than "pure investigation," and had almost total responsibility for daily classroom activities.

Toulmin's Model-Global and Local Arguments

Significant research has been done in the past decade drawing upon Toulmin's (1969) model of argumentation to analyze student proof and justification in mathematics class (Krummheuer, 1995; Krummheuer, 2007; Weber, Maher, Powell, & Lee, 2008; Inglis, Mejia-Ramos, & Simpson, 2007; Yakel, 2001). In particular, these researchers have used Toulmin's model to analyze and explain first-grade classroom discussion of number decompositions (Krummheuer, 2007), sixth-grade discussion of warrants in statistical arguments (Weber, et al., 2008) and a high school geometry class (Knipping, 2008). Krummheuer's (2007) goal was to describe the structure of the argument produced in classroom interaction and the different ways students were involved in the production of that argument. Knipping (2008) extended Krummheuer's work to encompass more sophisticated mathematical work. Yet, even then, Knipping made no effort to link possible student learning to classroom activity. Inglis, Mejia-Ramos, and Simpson's (2007) work suggested that analysis of mathematicians' proof production requires drawing upon Toulmin's full model as opposed to the reduced version more commonly used in mathematics education research. Toulmin's scheme has six basic types of statement, each of which play different roles in arguments. These are:

- The data (D): the foundation upon which the argument is based
- The conclusion (C): that which is being argued
- The warrant (W): justifies the relationship between the data and the conclusion
- The backing (B): supports the warrant by suggesting why it is valid, or, put another way, explains the permissibility of the warrant.
- The modal qualifier (Q): expresses a degree of confidence of the conclusion
- The rebuttal (R): states conditions under which the conclusion would not hold.

Traditional instructors typically model the mathematical behavior that they expect from their students (Fukawa-Connelly, 2010). In this case, the behavior is limited to proof-writing skills, including the level of detail that the instructor believes to be appropriate. Taken seriously, the idea that instructors model the level of detail they want students to include in arguments suggests that what the teacher does in class serves as a model for the level of detail that the students are to include in their written proofs. I propose that this includes whether, and when, proofs include explicit statements of warrants, backing, qualifiers and rebuttals. This paper will make explicit one connection between instruction and student work, but it does not make any effort to comment on the appropriateness of the instruction or the instructional goals. It should not be read, in any way, as a critique of the instructional methods. It is meant to provide a means to analyze teaching and student work and give an explanatory link between them.

Data and Methods

3.1 The teacher and institution

When the study began, Dr. Tripp (a pseudonym) was an assistant professor working towards tenure at a mid-sized doctoral granting institution in the Midwest. Dr. Tripp holds a doctorate; her research specialty is in algebra. She had taught abstract algebra a number of times prior to the study. In the context of this study, Dr. Tripp's most important characteristic is her self-description as a traditional teacher of abstract algebra. For her, this meant that lecturing in front of the class would be the predominant method of instruction, that "proofs form the backbone of this course," and that her organization and presentation of material would closely follow that of the text, *Abstract algebra: An introduction* (Hungerford, 1997). In other words, on the continuum of teaching from pure telling to pure discovery, Dr. Tripp claimed that she would be closer to pure telling.

3.2 The class

The class met four times per week for 50-minute sessions in the spring semester. Students had frequent homework assignments from the text, two in-class exams, and one final exam. The course featured a significant amount of ring theory, starting with the definition of a ring, moving onto quotient rings, and culminating with the construction of roots of irreducible polynomials. The class transitioned to the study of group theory with approximately one week remaining in the semester.

3.3 The students and data collected from students

Typically, the students were juniors who had completed a two-semester calculus sequence. All studentswere required to complete an introductory course on mathematical proof as prerequisite to abstract algebra; thirteen such students were enrolled in the course simultaneously in the section studied, including one sophomore and one senior. Twelve students agreed to participate passively in the study by allowing recording, transcription and analysis of their classroom activities. Five students actively participated by completing three written assessments. The first was a survey that asked them to summarize their mathematical preparation; this was followed by two content surveys, the first of which will be discussed in this study.

3.3.1 The students of the class.

There were five students from the class who agreed to actively participate in all aspects of the study (all identified via pseudonyms in work below). All of the students who agreed to participate had an overall GPA ranging from 3.6 to 3.9, which was representative of the students enrolled in abstract algebra that semester, according to the faculty. Four were junior mathematics majors, one was a mathematics education major and two also had a second major. All had earned either an A or B in their introduction to proofs class. However, the faculty did not believe it had been an effective course. Thus, I argue that it did not have a significant influence on the students' proof-writing. Finally, one active participant was a sophomore who was also doing independent research in graph theory with another professor in the department.

She had won the departmental freshman/sophomore prize in mathematics as both a freshman and a sophomore.

3.3.2 Content assessment instrument.

The content assessment instrument analyzed in this paper was given mid-semester. It presented a single task that assessed whether the participating students were able to determine if a proposed set with associated operations was a ring, which is a typical exercise in an introductory algebra course. The task, as given to the students, is shown in Figure 1.

Let $(\mathbf{Z}, +, \times)$ be the ordinary ring of integers: $\mathbf{Z} = \{\ldots -2,\ -1,\ 0,\ 1,\ 2,\ \ldots\}$, $n + m$ is the sum of n and m, $n \times m$ is the product of n and m. If n and m are in $\mathbf{Z}$ then $n \geq m$ if and only if $n - m$ is either positive or 0.

The algebraic system $R = (\mathbf{Z}, \oplus, \otimes)$ consists of the set $\mathbf{Z}$ with the two binary operations $\oplus$ and $\otimes$. The operation $\oplus$ is ordinary addition: $n \oplus m = n + m$. The operation $\otimes$ is defined by: $n \otimes m = \begin{cases} n & \text{if} \quad n \geq m \\ m & \text{if} \quad m \geq n \end{cases}$. Otherwise put, $n \otimes m$ is the maximum of n and m.

You are to determine if R is a ring. If it is a ring, show that it satisfies all of the defining properties of a ring. If it is not a ring, identify a needed property that R fails to satisfy, and demonstrate that this property is not satisfied by R.

Figure 1. Task given to students used in analysis

3.4 Methodology for analysis

I observed 18 of the class meetings, taking detailed field notes, and made video recordings of 15 of those classes. I transcribed all dialogue as well as all text on the board. The video camera primarily recorded the instructor, as she was the principle focus of all class activities. I reviewed all classroom video recordings and made a log of all episodes that included proof-writing or presentation. In all of the examples described below, pseudonyms are used to preserve the anonymity of the participants. These pseudonyms are consistent throughout the paper; all work and comments by Jeff, for example, are from the same person. In the transcript of classroom data, most student utterances are simply denoted via **S**, without regard for the student speaking, while all of Dr. Tripp's begin **Dr. Tripp.**

Criteria for proof-production or presentation was straightforward; an incident was logged as such when any member of the class community wrote or showed a formal mathematical proof that drew on symbolic notation and logical reasoning. This excluded any informal justification that was offered if it was not at that moment connected to formal proof. Work with particular examples of structures was categorized as proof as long as algebraic techniques were being used. I was inclined to count written work alone as modeling of proof-writing, simply because it was more likely to be copied into students' notes, but have also included some analysis of the spoken modeling of argumentation.

After identifying proof-writing episodes in class, I noted the following pair of proofs demonstrating that a function, $\overline{f}$, is a homomorphism of rings. The first proof was written by Dr. Tripp and the second by a student, Jeff, immediately after Dr. Tripp wrote her proof.

Dr. Tripp's work	*Jeff's work*
$\overline{f}((r+\ker f)+(t+\ker f))$	$\overline{f}((r+\ker f)(t+\ker f))$
$=\overline{f}((r+t)+\ker f)$	$=\overline{f}((rt)+\ker f)$
$=f(r+t)=f(r)+f(t)$	$=f(rt)=f(r)f(t)$
$=\overline{f}(r+\ker f)+\overline{f}(t+\ker f)$	$=\overline{f}(r+\ker f)\overline{f}(t+\ker f)$
$\therefore \overline{f}\ \ preserves\ \ addition$	$\therefore \overline{f}\ \ preserves\ \ multiplication$

That Jeff had perfectly reproduced Dr. Tripp's work was immediately apparent; it seemed obvious that he was using Dr. Tripp's proof as a model for his own. That is, I believed that Jeff had appropriated the manner of writing proof that Dr. Tripp had modeled in her work.

Similarly, Dr. Tripp wrote out a proof demonstrating that a function preserves addition, then called Nathan to the board. He wrote a near-perfect copy:

Dr. Tripp's work	*Nathan's work*
$f(a,b)+f(c,d)$	$f(ac,bd)$
$=b+d$	$=bd$
$=f((a+c,b+d))$	$=f(a,b)f(c,d)$

It is important to note that these are the only two cases where Dr. Tripp wrote an argument and then had a student write an argument about a similar statement. There were other cases where students gave proofs at the board or using an overhead transparency, but none were similar to Dr. Tripp's work.

Thus, to pursue the question of whether students had appropriated proof-writing behavior that Dr. Tripp modeled required analysis of the proofs she wrote during class and of the students' work outside of class. She, collectively with the students, wrote one complete proof of the fact that a structure is a ring, verifying each of the required properties and then showing that the ideal property holds. She wrote one proof showing that a structure is a subgroup, three proofs demonstrating that a function is a homomorphism, and proofs that the kernel is a given set, that a function is onto and that a function is one-to-one. Additionally, I observed her or a student present five property-verification arguments: that a given function is well-defined, that the quotient structure R/I is associative under addition, that multiplication distributes over addition when R is a ring and I is an ideal of the ring, that all non-zero elements of a given ring have multiplicative inverses, and, finally, that a given function is onto a specified domain. Finally, I observed 6 presentations by students that were prepared ahead of time, which essentially were proofs copied from the text and thus limited in their usefulness to the present analysis.

In sum, I observed five arguments that required a number of independent properties had to be demonstrated in order to show that a structure is as desired. I also observed 23 individual property verification arguments across these reserve-structure proofs and the other arguments described above. Because I only have two examples of students writing proofs involving homomorphism property verifications, both shown above, I decided to concentrate my analysis of Dr. Tripp's teaching on ring and group property verification arguments.

I analyzed all arguments using Toulmin's (1969) model, noting both when an aspect of argumentation was written and when one was spoken aloud. In this paper I Dr. Tripp's demonstration of all of the properties required for a sub-ring proof.

All of Dr. Tripp's written and spoken statements used in proof-writing were classified as one of the following: data, warrant, backing, qualifier or conclusion. The students' written work was similarly classified. A statement was classified as data if it directly followed "if" or "let" or was stated as an obvious fact without support (such as $f(0)=0$ when discussing a homomorphism, because this is an elementary fact about morphisms). A statement was classified as a conclusion when it followed "then" or when it was presented as a deduction from a piece of data, such as "R fulfills X property." A statement was classified as a warrant when it linked the data and conclusion in a way that explained how the data supported the conclusion and drew upon previously demonstrated facts, or upon facts that had been stated as part of the hypotheses (for instance, information about elements included in sets). A statement was classified as backing if it explained the applicability of a warrant such as the claim "We have previously proved…" Finally, a statement was classified as a qualifier if it limited the generalizability of the statement, provided a possible exception to the statement, or asserted facts about logic that served a similar purpose. In complex proofs some statements are first introduced as conclusions and then reused as data for a subsequent argument.

Dr. Tripp described herself as a traditional, lecture-based teacher, but it is important to note that in her classroom, the students were likely to participate in proof-writing or presentation activities. In 29 observed proofs, the students presented seven by themselves. Dr. Tripp used participatory proof-writing schemes 21 times, engaging students in a question-and-answer dialogue as she was working. Only once did she deliver the caricatured "lecture" where students sit quietly and take notes without questions or comments while the professor presents the proof. Thus, her classroom was filled with dialogue, and, in order to analyze her presentations of proofs, it is necessary to recognize that this dialogue plays an important role in her pedagogy. Thus, I created a summary table that captured how frequently in these arguments Dr. Tripp spoke, wrote, or both spoke and wrote or asked a student to articulate a particular aspect of an argument. I employed a similar methodology to analyze the students' written work.

Analysis

4.1 Analysis of teaching property verification arguments

Property verification arguments were more readily accessible for analysis as Dr. Tripp always wrote out substantial portions of them. Similarly, there were far more of them than there are top-level arguments because each proof about a given structure required a top-level argument and multiple property-verification arguments (for example, demonstrating that a structure is a ring requires a top-level argument that coordinates six property-verification arguments). Below is Dr. Tripp's written proof that the kernel of a ring homomorphism is an ideal:

K = {m| f(m) = f(0)}
We already know this is an ideal!
$f(0_R) = 0_S,\ 0_R \in K$
$\forall a,b \in K,\ f(a+b) = f(a) + f(b) = 0_S + 0_S = 0_S$
$a+b \in K$
$f(ab) = f(a)f(b) = 0_S f(b) = 0_S$
$f(ra) = f(r)f(a) = f(r)0_S = 0_S,\ ra,ar \in K$
[*Dr. Tripp erased the previous work and began writing on a clean board*]
$\forall a \in K,\ f(-a) = -f(a) = -0 = 0$
K is an ideal. [*Then underlined*]

There are five separate property-verification arguments represented in Dr. Tripp's proof: additive identity and non-empty (done together), additive closure, multiplicative closure, the ideal condition, and the existence of additive inverses. The additive identity proof consists of only two clauses: the data, "$f(0_R) = 0_S$;" an unwritten warrant, "We have proved that a homomorphism maps the identity to the identity;" an unwritten backing, "0_R is the additive identity element of *R*;" and a written conclusion, "$0_R \in K$," which corresponds to a stated but unwritten conclusion, "So, it's got something in it." This argument is diagramed in Figure 2.

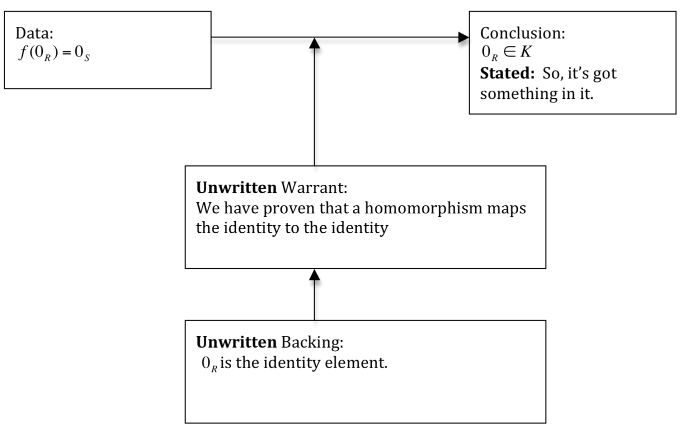

Figure 2. Dr. Tripp's non-empty argument.

Her subsequent arguments had a similar relationship between written and stated elements. In her proof that the set K is closed under addition, Dr. Tripp wrote:

$$\forall a,b \in K,\ f(a+b) = f(a) + f(b) = 0_S + 0_S = 0_S$$
$$a+b \in K$$

The statement $\forall a,b \in K,\ f(a+b) = 0_S$ is data, $f(a+b) = f(a) + f(b) = 0_S + 0_S = 0_S$ is a written warrant, and $a+b \in K$ is the written conclusion. Dr. Tripp did not write any backing for her warrant, but while writing out the proof, engaged in the following dialogue with a student:

Dr. Tripp: Ok, so, let's look at what *f* does to *a* + *b*. So, S, what can I say about *f*(*a* + *b*)?

S: It equals $f(a)$ plus $f(b)$.
Dr. Tripp: Is there anything I know about $f(a)$ now?
S: It equals zero.
Dr. Tripp: Great, and $f(b)$? And what do I know about zero plus zero? That's zero. Great. So, $a + b$ meets the condition it needs to be in K. [pause] So, that's the property of f preserving addition that we just used, and that gives us that the kernel is closed under addition.

That is, Dr. Tripp stated, but did not write, two different backing statements for the written warrant. She first explained why it is permissible to rewrite $f(a+b)$ as $f(a)+f(b)$ and explained why $f(a)$ and $f(b)$ were both equal to the additive identity element in the ring S. We diagram her argument below.

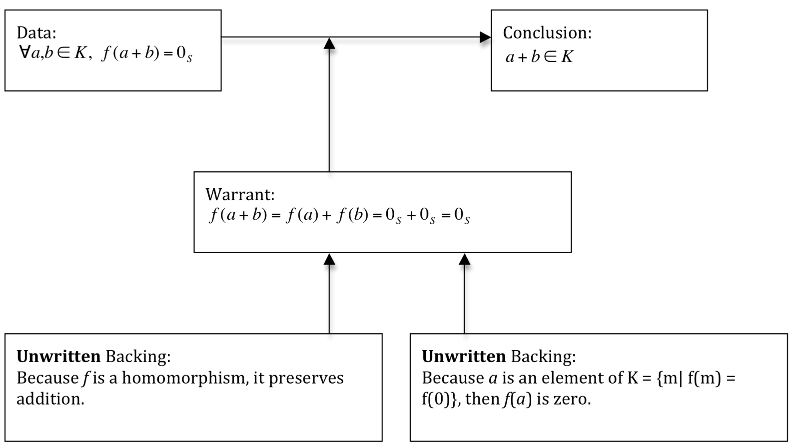

Figure 3. Dr. Tripp's closure argument

Dr. Tripp presented a written piece of data, conclusion and warrant. She stated, but did not write the backing that she provided. Analysis of the remainder of the property verification arguments that make up this proof show the same pattern; backing is stated, not written, while the rest of the argument is written.

To summarize Dr. Tripp's presentations in all of her proofs, data and conclusions are (almost) always written while the warrants were generally unwritten and included no qualifiers. However, the warrants were generally stated aloud during the classroom discussion, either by Dr. Tripp or by a student.

	Data	Warrant	Backing	Qualifier	Conclusion
Written	65	14	5	0	64

Table 1. Dr. Tripp's written argumentation

When we include her commentary in the argument analysis for the proofs presented above, we see the following:

	Data	Warrant	Backing	Qualifier	Conclusion
Written or stated	65	35	15	0	64

Table 2. Dr. Tripp's argumentation

That is, Dr. Tripp only infrequently wrote the warrant, and almost never wrote the backing, but modeled both fairly frequently in spoken form.

Given this pattern, we can expect to see similar trends in student proof-writing. We expect to see that students always or almost always wrote the data and conclusion, because Dr. Tripp always wrote those. But her mixed writing of warrants may not have provided her students a consistent model for their own work; thus, we can expect missed results in the written arguments given by the students. Although, we would expect the students to write warrants at a higher rate that Dr. Tripp did because of her frequency of stating them aloud. This is because Dr. Tripp shows a pattern of mixed written and spoken warrants when modeling proofs. Finally, we should expect that the students never, or almost never, write a backing or qualifier in their proofs, since Dr. Tripp never wrote either of these.

4.2 Analysis of student work

Taken together, the students made 33 different property arguments, which I analyzed using Toulmin's scheme. I will present the data on student arguments in three different sections, in the following order: additive property arguments, multiplicative property arguments, and the distributive property. I will show a representative sample of each and summarize the presence or absence of the different elements (data, warrant, backing, qualifiers, conclusion) in the students' written work. I will finish by summarizing the written elements of the students' arguments and comparing those with Dr. Tripp's modeled behavior.

4.2.1 The additive property arguments

Four of the students wrote proofs of the additive properties of closure, identity, inverses, and of the commutative and associate properties of all elements. When I analyzed these arguments using Toulmin's model, there were all essentially the same. The argument was given by the following relationship:

Data: Addition on the integers fulfills this property

Warrant: The operation $\oplus$ on this ring is defined as standard arithmetic on the integers.

Conclusion: R fulfills this property.

I provide an example of such an argument below to demonstrate that the ring has an additive identity element.

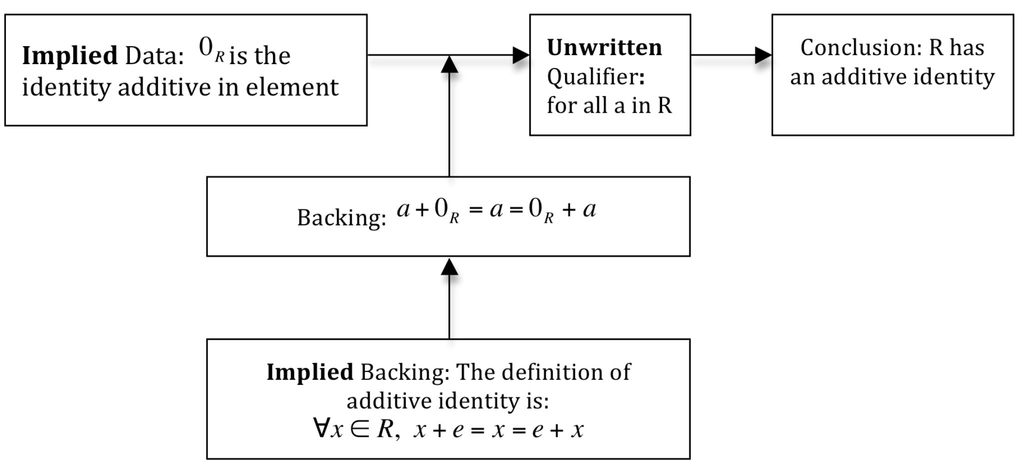

Figure 4. An example of a student argument about additive identity

This argument structure may be understood as characteristic of their work on addition properties. One student went so far as to summarize all the property arguments about addition at one time using the structure above. This student did not add any qualifiers and did not give any explicit backing. The table below summarizes the students' work on their proofs of additive properties:

	Data	Warrant	Backing	Qualifier	Conclusion
Stated	19	4	1		19
Unstated but implied	1	16			1

Table 3. Student's additive argumentation

In the proofs of these additive structures the students almost always stated the data and conclusion and only stated a warrant four out of sixteen times, a 25% rate. This low rate may be due to the fact that the students felt the additive properties of the integers were nearly self-evident by the middle of the semester (after 8 weeks spent studying rings) and thus, did not require a warrant or backing. The students typically wrote lengthier arguments in proofs of the multiplicative properties.

4.2.2 The multiplicative property arguments

The assessment required only that the students demonstrate that the defined structure was a ring, and, as a result, the students only attempted arguments that the defined multiplication was closed and associative. Because there were fewer properties to verify, there were fewer arguments to analyze. Moreover, possibly because the operation was less familiar, there was more variation in the structure of the arguments that the students wrote. For example, the students' arguments were more likely to state a warrant. One argument given was invalid. Multiple students included stated backing, and one student used a qualifier statement.

Let us begin by considering an argument for the closure of R under multiplication that includes a written warrant, unwritten backing, and a written qualifier. Jeff wrote:

(vi) If $a, b \in R$, then ab is either a or b, depending on which is larger. Either way, $ab \in R$ and so R is closed under multiplication.

A diagram of Jeff's argument using Toulmin's model:

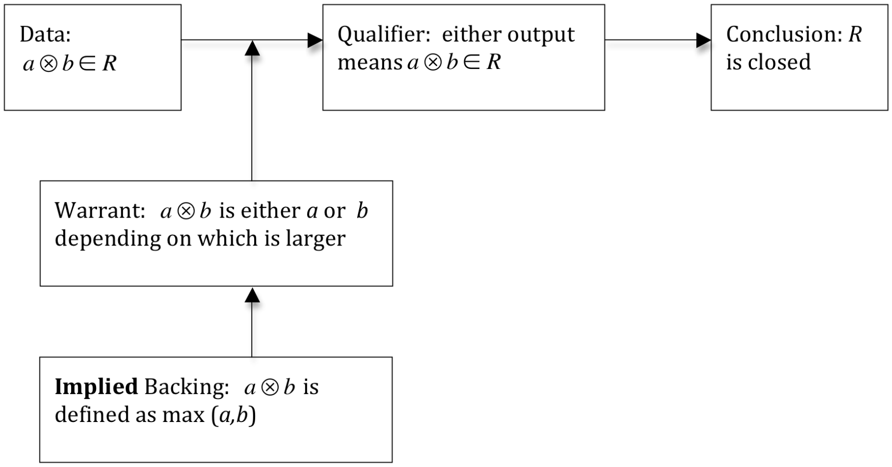

Figure 5. Jeff's argument for closure under multiplication

Aside from the written qualifier, this argument is illustrative of the closure arguments that the students wrote.

In all three of the arguments we have seen so far the student wrote at least one warrant. The table below summarizes the number of times that the students left unstated, but implied, different aspects of the argument about multiplicative properties.

	Data	Warrant	Backing	Qualifier	Conclusion
Stated	8	6		1	6
Needed but missing		2			

Table 4. Student's multiplicative property argumentation

This table suggests that Nathan, Jeff, and Aurora's work was reasonably representative of the class in terms of the elements of the argument that they wrote or left implicit (excepting Jeff's inclusion of a qualifier). In the examples above, the students wrote the data, the conclusion, and the warrants, but not the backing. This is reflected in the students' written work where we see that, generally speaking, they wrote the data, the warrant and the conclusion, but not the backing. This mirrored their work with the additive properties. The students wrote warrants at a higher rate than for the additive property arguments. This difference may result from the difference in the students' familiarity with the operations; the first operation of the proposed structure is very familiar to the students, standard integer addition. The second operation, the maximum operation, is less familiar to the students and therefore, they may have believed that this unfamiliarity meant that they should include more detail in their written proof.

4.3 Summary:

All of the student data on property-verification arguments is aggregated in the table below, without reference to the validity of the student's claim (one student made an invalid claim about distribution).

	Data	Warrant	Backing	Qualifier	Conclusion
Stated	32	14	2	1	30

Table 5. Student's argumentation

A few observations can be made. First, the students almost universally wrote both the data and conclusion of their arguments. They were much more varied when writing out (or implying) warrants in their work; of the 33 written arguments, only 14 included a written warrant. Second, the students almost never wrote out a backing or a qualifier, writing only two backings and one qualifier in 33 arguments. Yet, there was a difference in the rate at which the students wrote details between the purely additive arguments and those that involved the other operation the, maximum operation. The students wrote more details about the less familiar maximum operation in their proofs.

In sum, we see exactly what we expected from the students given Dr. Tripp's modeling of written proofs. She always wrote the data and conclusion, hardly even wrote a warrant (though she did say them along approximately half the time), and never wrote a backing or a qualifier. Clearly, in this case, the students performed the same way: they consistently wrote the data and conclusion, wrote a warrant in just under half of their arguments, and very rarely (6% of the time) wrote backing or a qualifier (3% of the time). In other words, the students seem to have appropriated Dr. Tripp's proof-writing behavior; they gave similar levels of argument detail in their written work.

5. Significance and directions for further research

This article makes three significant contributions. First, it shows one way to draw upon Toulmin's (1969) model of argumentation to analyze proof-presentation in undergraduate courses, including traditionally taught abstract algebra courses. This process required analysis of what the instructor wrote and of the classroom dialogue surrounding the proof-presentation. In both cases, Toulmin's model helped explain the relationship between the instructor's written proof and the classroom dialogue that she led. When Dr. Tripp's presentation is understood to model the type of mathematical behavior that she wants her students to demonstrate, we may conclude that she wants her students to always be able to articulate the data, warrant and conclusion of an argument. She also modeled use of backing and qualifier statements by stating them aloud, rather than writing them down. Dr. Tripp also wrote out warrants for her data slightly less than half of the time, but always spoke them. Thus, I demonstrate through my analysis that instructors may be inconsistent in the level of detail that they include in written arguments.

Second, I analyzed student work using Toulmin's (1969) model for argumentation. Taken as a whole, the students' collective proof-writing was fairly consistent. They almost always wrote the data and conclusion and avoided any written backing or qualifiers. Moreover, for the additive property verification tasks, the students essentially avoided writing out warrants, whereas on the multiplicative and distributive property verification tasks, they generally wrote the warrants.

Finally, I showed a link between Dr. Tripp's modeling of mathematical proof-writing and the students' demonstrated proof-writing. When I analyzed the students' work on property-verification arguments, I found that they wrote using a level of detail similar to that modeled by their instructor. The students were observed to write data, conclusions, warrants, backing and qualifiers at nearly the same rate that had been observed in Dr. Tripp's modeling of proof-writing. Furthermore, as noted above, Dr. Tripp was inconsistent in writing versus speaking aloud warrants, and we see that the students were similarly inconsistent, writing a warrant in slightly less than half, 42%, of all of their arguments. That is, the students almost perfectly demonstrated that they had appropriated Dr. Tripp's modeled proof-writing in terms of the level of detail that they included in their written work.

This research immediately suggests two future directions for research, both directed towards better understanding the development of students' mathematical proficiency. First, the use of Toulmin's framework to analyze teaching was helpful in making sense of some aspects of Dr. Tripp's writing and her classroom dialogue, but cannot explain all aspects of her modeling of proof-writing. We need significant research that studies the teaching of proof-writing (Harel & Sowder, 2007; Harel & Fuller, 2009), and, in particular, lecture-based teaching of proof-writing. Furthermore, we need new theoretical lenses to make sense of what lecture-based teachers are doing in classes that provide a means to explain student mathematical proficiency. In this vein, it is helpful to distinguish between introduction to proof courses and a proof-based content courses. A proof-based content course has the dual goals of improving the students' ability to write and critique proofs as well as increase their proficiency with the particular mathematical content of the course. In contrast, an introduction to proof course has one goal: to teach students the logic and mechanics of writing proofs. Thus, theoretical models focused solely on the teaching of proof-writing or proof-comprehension will be insufficient if they cannot help researchers account for instructors' attempts to teach content, as well.

Thinking of Dr. Tripp as modeling behavior that she wanted students to adopt was helpful in understanding her classroom behavior. Thus the idea that teachers consciously model appropriate mathematical behavior for their students is worth pursuing as a means of making sense of lecture-based undergraduate courses. For example, we should explore how instructors model other fundamental mathematical skills for their students, such as definitions and examples, organization and connection of knowledge, and abstraction and generalization from examples, to name but a few. It is important to note that this list is suggested by the habits that mathematicians believe are important in the practice of mathematics and may not completely capture the list of behaviors that instructors actually model for their students. It almost certainly does not explain what is presented as the *work* of doing mathematics—the research methods of mathematicians, their habits and practices. Significantly more research is needed to explain traditional lecture-based instruction of proof-based content courses in general, and abstract algebra courses in particular.

Lastly, given a goal of improving student learning, this line of analysis offers one possible avenue for mathematics educators to work with mathematicians to make minor modifications of their teaching that may lead to increased student success. In this study I reported seemingly inconsistent behavior on the part of the instructor, which was linked to inconsistent behavior on the part of the students. We might suggest that classroom professors make explicit statements, such as, "Listen to the types of questions I ask you and ask myself. These are the kinds of things *you* should be asking while you write proofs." Or, "To decide when you should write down the answers to these questions you should …" This type of meta-dialogue could lead the students to a better understanding of when warrants and backing must be explicitly stated in a proof, as well as decrease the writing of invalid or incorrect proofs. Given that Dr. Tripp's class, and by extension, many other lecture-based classes, already include significant conversation and dialogue, this addition to the classroom discussion seems a relatively small change that may have a more significant effect on students' learning than a more difficult switch to an inquiry-oriented pedagogy.

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