ANALYZING THE TEACHING OF ADVANCED MATHEMATICS COURSES VIA THE ENACTED EXAMPLE SPACE

Tim Fukawa-Connelly, Charlene Newton and Mariah Shrey
The University of New Hampshire
Tim.fc@unh.edu

In advanced undergraduate mathematics, students are expected to make sense of abstract definitions of mathematical concepts, create conjectures about those concepts, write proofs and exhibit counterexamples of these abstract concepts. In all of these actions, students may draw upon a rich store of examples in order to make meaningful progress. We have drawn on the concept of an example space (Watson & Mason, 2008) for a particular concept. We adapted it and defined the concepts of example neighborhood, methods of example construction, and the functions of examples to create a methodology for studying the teaching of proof-based courses. We demonstrate our method via a case study from an undergraduate abstract algebra course.

Keywords: teaching, examples, definitions, proof, abstract algebra

Introduction

Studying teaching is by nature a difficult process. For this reason, "very little empirical research has yet described and analyzed the practices of teachers of mathematics" (Speer et al., 2010, p. 99) at the undergraduate level despite repeated suggestions for this type of study (Harel & Sowder, 2007; Harel & Fuller, 2009). That is, "researchers' questions, methods, and analyses have not generally targeted what teachers say, do, and think about collegiate classrooms in an extensive or detailed way" (Speer et al., 2010, p. 105). Although it is important, in and of itself, to better understand the reality of collegiate mathematics classes, it is also essential to develop research-based descriptions of traditional undergraduate classes to support and explain studies of student's mathematical proficiency.

Within the context of the advanced undergraduate mathematics classes one area of increasing emphasis is on the use of examples. First, research indicates that "exemplification is a critical feature in all kinds of teaching, with all kinds of mathematical knowledge as an aim." (Bills & Watson, 2008, p. 77). This has resulted in the study of example usage among students. For example, Alcock and Inglis (2008) examined doctoral students' use of examples in evaluating the truth-value of claims as a way to understand what students might do. Dahlberg and Housman (1997) found that students who generated their own examples were more likely to develop initial understandings of concepts, and Mason and Watson (2008) described ways to make use of the range of possible variation for pedagogical purposes, to name but a few. Yet, these studies are on students' uses of examples and there is no corresponding study of the teaching of proof-based classes and instructor's teaching with examples.

In the following article we discuss the current state of research on examples as tools for learning mathematics and draw implications for teaching. We then use those implications to outline a methodology for analyzing undergraduate mathematics teaching through the lens of examples. Finally, we show an example of using the method by analyzing the instruction of an introductory abstract algebra course In the case study that follows we describe the teaching of the class and then analyze the teaching via the lens of the enacted example space.

Literature Review

1.1 Studying teaching

At the undergraduate level, little is known about the processes of teaching and learning of proof-based mathematics courses. Mejia-Ramos and Inglis (2009) conducted a literature search and found only 102 educational research papers that studied undergraduate students' experiences writing, reading and understanding proof. Of those 102 papers, Mejia-Ramos and Inglis found no papers describing how students understand instructor presentation of proof. There is only one study specifically designed to investigate how instructors use examples to support their teaching of proof-based classes. There are four studies that give some description of how instructors use of examples in teaching.

The first was specifically drawing on student's use of examples as a way of understanding classroom activities. Larsen and Zandieh (2008) adapted Lakatos' (1976) lens of proofs and refutations from the way that mathematics might be created to describe how students might recreate mathematical ideas. They first described how examples are used to create and modify definitions, which they called "monster barring." Second, they described "exception barring" as a means of creating or modifying a conjecture. Finally, they described proof analysis as a further way to refine a conjecture and generate new concepts. At each step they showed evidence of students using examples in the specified way drawn from the context of an inquiry-oriented abstract algebra class.

Weber (2004) analyzed the teaching of a real analysis class with a focus on how the instructor presented proofs. He identified three basic styles of teaching proof: *logico-structural*, *procedural*, and *semantic*. The first two styles specifically avoided any use of examples or diagrams. The last, the *semantic* teaching style, is characterized by the instructor's use of intuitive descriptions of concepts and relationships. In each case, the instructor's teaching style was chosen to help students acquire some specific type of knowledge or ability needed to construct proofs (as well as to demonstrate a large number of similar proofs). The semantic style was meant to help the students learn how an understanding of the concepts of mathematics, when supported by procedural fluency with the logic and language of mathematics, would support proof writing. An example of a proof-presented in this style that includes example usage would be one where the instructor divides the board into halves and presents the proof on one half while showing each part in the case of a specific example on the other half. The instructor specifically said that the goal was for the students to "have rich imagery that they could associate with the concepts being taught" (Weber, 2004, pp. 126-127).

While the instructor in Weber's paper claimed that he only used semantic style proofs at the end of the course, many mathematicians claim that examples are important both to their own work and to their teaching. Alcock (2009) and Weber (2010) each interviewed a collection of mathematicians about their teaching practices that support the learning of proof. The two groups of mathematicians interviewed described many similar practices and both groups emphasized the importance of teaching students to instantiate by assigning students problems that require example generation (Alcock, 2009). Similarly, eight of the nine mathematicians in Weber's (2010) study said that they normally accompany a proof with an example or draw a diagram that is intended to support students' understanding of the proof. Given that examples are believed to support students' developing understanding of concepts and proof-writing abilities, it is important to better understand why examples are important in mathematics and the learning of advanced mathematics.

1.2 What are examples, and, what will we study?

This paper draws upon Watson and Mason's definition of an example as "any mathematical object from which it is expected to generalize" (2005, p. 3). All further discussion of the role and importance of examples will rely upon an important pedagogical distinction between examples of a concept (such as group, ring, field) and examples of a process (such as using the Euclidean Algorithm to determine the GCD). While both can be understood as mathematical objects from which to generalize, the paper that follows will draw upon examples of concepts in mathematics teaching, though we assert a similar methodology could be used to study examples of processes. Examples of concepts are uniquely powerful in both mathematics and the teaching of mathematics and, as a result, there are numerous reasons to study the use of examples in classes.

1.3 Why examples are so important in advanced mathematics and mathematics teaching

Mathematicians and students use collections of examples as references to develop intuition and as a means to generate, test, and refine conjectures (Alcock & Inglis, 2008; Michener, 1978). When a mathematician comes upon or creates a conjecture, hypothesis, or theorem that is not obviously true, Courant (1981) claimed that the mathematician's first reaction was to call upon an example so as to think about the general through a particular case. Goldenberg and Mason (2008) stated that there is little difference between examples and counterexamples when testing and refining conjectures; the difference lies in what the reader attends to. In other words, an example that demonstrates the truth of the theorem can be considered a non-example of an incorrect version of the theorem, and a counterexample to a proposed theorem allows for reformulation of an incorrect claim into a correct claim. In either case, the purpose of the example is to provide a more familiar and concrete means to explore ideas and to check the conditions of and evaluate constraints in theorem formulation.

These claims for the mathematical importance of examples have also shaped the way that examples are used in teaching. Examples are often used to introduce and motivate topics in class. In particular, examples that introduce concepts give individuals a concrete and potentially familiar means to explore and understand the constraints and affordances of a definition. Dienes (1963) argued that mathematics learners require at least three examples of a concept in order to develop understanding. That is, examples give the student a collection of familiar objects from which they can develop abstractions that are eventually reified into concepts (Sfard, 1994). Similar to Goldberg and Mason's (2008) claims about non-examples helping individuals to understand the conditions of a theorem, Zazkis and Leiken (2008) refer to pertinent non-examples in the situation of testing conjectures for truth-value because non-examples also help mathematics learners make sense of particular aspects of definitions by asking, "Why does this not satisfy the definition?" Thus, examples and pertinent non-examples give learners more opportunities to gain a complete understanding of each clause of a given definition, both of what that definition allows and of what it prohibits in terms of structures. Moreover, as Goldenberg and Mason posited, it is by exploring examples that "learners encounter nuances of meaning, variation in parameters and other aspects that can change" (2008, p. 184).

While examples and non-examples that students encounter from the curriculum are important in helping them to develop an understanding of important concepts, Zazkis and Leiken (2008) also describe the importance of learner-generated examples. They claim that learner-generated examples are especially for instructors. Instructors can evaluate their students' understanding of the definition by asking them to construct examples. Therefore, they can be used to diagnose problematic aspects of the student's understanding. Thus, examples serve as a type of formative assessment that allows faculty to propose structures (either pertinent non-examples or examples) to better help the students develop their understanding of the concept.

In summary, examples can be used in mathematics and mathematics teaching for a number of different functions. Lakatos (1976) described the uses of examples as articulating and refining definitions and conjectures, that is, as objects from which to generalize. Mathematicians use examples to attempt to check the validity of conjectures and as clues to the proofs of theorems. In teaching, examples provide a way for students to attach meaning to definitions (Goldenberg & Mason, 2008). Similarly, they can also help develop understanding of proofs for appropriate theorems (Alcock, 2009; Weber, 2010). We believe that the range of functions that students are exposed to is important in their understanding of the field of mathematics. Essentially, students should see a wide range of examples functioning in different ways: exemplifying definitions, helping to generate and test conjectures and as providing clues to proofs.

Mason and Watson (2008) have taken the mathematical and pedagogical purposes of examples and created a description of a construct that mathematics users might have. They described the *example space* as a construct that allows mathematics users to develop an appropriate sense of the concept being exemplified and to learn to adequately perform all of these other functions of examples.

1.4 Example spaces: Students' range of thought, knowing what can vary, knowing what must stay the same

An example space is the "experience of having come to mind one or more classes of mathematics objects together with construction methods and associations" (Goldenberg & Mason, 2008, p. 189). It may include relatively frequently accessed members of the class, less accessed members, and new members (via construction methods). Two features of example spaces that Mason and Watson (2008) called important: what aspects of the examples the learner realizes can be varied, and what range of variation the learner believes is appropriate.

Goldenberg and Mason (2008) draw upon variation theory and claim that learners need experience with many examples. The first examples that learners experience are particularly important as they are often the ones that students most closely link with the concept, and, as a result, should be very carefully chosen (Zodik & Zaslavsky, 2008). Students may modify their understanding of the definition of the concept based upon their image of the concept (Vinner, 1991). As a result, when students experience an early example or set of examples that they adopt as their concept image, it can shape their understanding of the concept itself. Thus, the early examples need to help students develop a complete concept image with all appropriate complexity. Vinner suggests that "only non-routine problems, in which incomplete concept images might be misleading, can encourage people to" develop more appropriate understandings (1991, p. 73).

Golderberg and Mason (2008) further the claim that the learners need to work with a carefully chosen set of examples closely in time. They believe that this will allow learners to "locate dimensions of possible variation… [and] discern which aspects can vary and which are structural" (Goldenberg & Mason, 2008, p. 186). Dienes (1963) argued students should encounter sets of examples that are narrowly constructed so that examples within each set vary along only a very limited number of dimensions. Thus, we believe that students should work with multiple sets of examples, each with variations along different dimensions so that they are able to apprehend both what aspects of an example can be varied and the range of possible variation for each aspect.

If we believe the arguments put forth about the importance of showing students a range of examples with different variation in carefully constructed ways, it follows that examining the collection of examples of a certain topic that students are exposed to and the order in which they encounter them could allow insight into the types of understandings that students will develop in terms of what can be varied and the range of possible variation. Thus, the range and organization of examples is directly linked to possible student learning. Of course, mathematics learners can construct appropriate or inappropriate beliefs from the collections of examples.

For example, when learning the definition of a group, the possible aspects of a group include characteristics of the underlying set, the group itself, and of the behavior of specific elements. In particular, we would want a learner to believe that the size of the set of a group is immaterial to the definition. In terms of the range of possible variation, we would want the learner to recognize that the cardinality could vary from 1 to groups with infinite cardinality.

A final important feature of an example space, in the way that Goldenberg and Mason (2008) have defined it, is that it purposefully includes construction methods and associations, such as links to important theorems and relations to other constructs. These links allow mathematicians and mathematics learners to create new examples that meet specific criteria of theorems and to discover which classes of objects are most relevant in particular situations.

Our Theory of Assessing the Enacted Example Space

We have drawn on the work in Section 1.4 to articulate a method for examining the enacted example space that uses three filters to describe the set of examples. We call these filters: (1) *example neighborhood*, (2) *example construction*, and (3) *the function of the example*.

First, we define the *example neighborhood* as the entire collection of examples that the students are exposed to during the course of their studies of a particular construct. These may be concrete examples or relevant non-examples of a given concept. A typical *example neighborhood* is the sequence of examples given to support the definition of a concept. We call this type of neighborhood a "definition-example" neighborhood. Another type of neighborhood arises in student-worked problems initiated with a stem such as, "Determine whether the structures below are examples of a _________."

We analyze how the examples in the example neighborhood are organized on four levels: (1) what is the first example given, (2) what examples are near the concept temporally, (3) what is the range of permissible variation that students experience, and (4) what variation constraint do students experience. We pay particular attention to the first few examples, as instructors believe they are often the ones that students most closely link with the concept (Zodik & Zaslavsky, 2008). Examples have less immediate relation to the central example(s) when the further they are removed either temporally or conceptually, depending on the number of aspects that vary from the central example(s). In terms of the range of permissible variation we assess what examples the students are given access to and the ranges of variation that those encompass. Similarly, we look at the set of non-examples that the students have access to and how those non-examples limit the concept in question. In assessing the variation constraint, we adopt Dienes (1963) argument that students should see examples that vary only in a constrained manner so that they can determine what is structural and what is allowed to vary, as well as apprehend the range of permissible variation. Then, he argued, they should see other examples that vary along a different dimension. As a result, we argue that early examples that vary along too many dimensions may actually lower the potential value for student learning. Similarly, a collection of examples that fails to support student construction of critical aspects of the concept will also lead to lessened possible student learning.

Secondly, we examine *example construction* to support a particular concept. Example construction focuses on the range of possible variation included in the neighborhood of a particular example space. The analysis of example construction focuses on how examples are created and the tools for creating additional examples. Example construction also allows mapping from concrete examples to a broad description of the example space that students may be able to populate themselves. In this way, the example space explicitly includes both examples and the means of construction (Goldenberg & Mason, 2008).

Third, we describe the *function of the examples* used in the classroom. We mean this in two different ways. First, we examine which examples are called upon most often. Frequently used examples may obtain "ready access" status for students (linked to Vinner's (1991) concept of *evoked concept image*). The frequency of use gives us a means to assess or predict the student's perception of the relative importance of each example and provides a way to predict which examples can most readily function as an example for the students. Secondly, we describe what the example was being used to do, that is, the mathematical intent of the example. Some possible functions of examples are exemplifying a definition, creating or refining of a definition (c.f., Larsen & Zandieh, 2008), articulating or exploring a conjecture, as well as illustrating a proof. We assess examples separately using each filter, and then read them together to analyze the example space. Taken together, these three rounds of assessment of the example space contribute to our proposed method of assessing the mathematical quality of instruction at the advanced undergraduate level.

We do believe that there is a possible balance that may be achieved by professors in giving the students the ability to populate their example spaces. We can imagine an instructor that directly gives the students access to a wide range of examples but few methods of construction. This range of examples would still give the students the possibility of developing a rich example space. At the other extreme, the instructor may give the students access to a relatively few examples, but a rich set of tools for example construction and a set of tasks that requires and supports the students in developing their own examples. We do not hold either of these extremes to be normative, but do believe that it is important that the students have the opportunity to develop a well-defined example space.

While it is true that the depth with which students engage the examples matters for their learning (Vinner, 1991), it is impossible to make any low-inference judgments about this from classroom observation. Students may be engaging with the material as it is presented in a lecture but, because they are not giving outward sign it is impossible to make inferences. Similarly, in an inquiry-based class, while students may be observed engaged in active discussions in small group, unless each group is monitored it is impossible to make judgments about their level of engagement. Other students presenting at the board is similar to the situation found in a lecture. Finally, even with suggested or required homework problems there is no way of evaluating the students' level of engagement without direct observation. Due to these measurement difficulties, in the method described below we make few claims about the student's level of engagement with the examples outside of class. We will distinguish between those problems that are suggested and those that are required, but it is certain that individual students will engage with the suggested problems in very different ways. As a result, we will make few judgments about the students' levels of engagement other than to distinguish cases where students were observed to actively engage with the examples from those where they were not.

Methods

3.1 Setting

In order to study the enacted curriculum, we collected data from an introductory abstract algebra class taught by a tenure-stream faculty member with an appointment in the department of mathematics and statistics at a mid-sized doctoral granting institution in the Northeast. The class was designed to be an introduction to the basic concepts of abstract algebra including groups, rings and fields. The class met three times per week for 70 minutes for 16 weeks. Students had frequent homework assignments, three in-class exams and one final exam.

The class had a mix of mathematics, mathematics education, and other STEM majors. Students ranged from sophomores to graduate students. The class had approximately 25 students. Although it was highly recommended that students had completed an introduction to proofs course, there were other ways to satisfy the pre-requisite (such as a lower-division linear algebra course) and as a result the student's prior proof-writing experience was quite varied.

3.2 Collecting data

In the lecture-based class we observed and video recorded 25 consecutive class meetings. We began with the class meeting immediately preceding the introduction of the formal definition of a mathematical group and ended with the definition of a factor group (quotient group). The camera was placed at the back of the class and pointed towards the board in order to best capture what the instructor said and what was written on the board. We also collected copies of all handouts given in class (homework assignments and exams) as well as tracked the assigned practice problems from the text.

All video data was digitized. Transana was used to code all incidents where an example or non-example was shown, constructed or analyzed in class. From the practice problems, homework assignments, and exams we recorded each instance where students were asked to work with a specific structure. Examples of problem-stems that indicated students were to consider a particular example were, "determine if the following form a group," or "Show that X is a group."

3.2 Analyzing data

We focused our analysis on the instructional uses of examples and non-examples of groups. In order to do this, we created an example log similar to Rasmussen and Stephan's (2008) argument log that included four columns to describe each example or non-example. The table was organized as follows:

- The first column listed each example or non-example of the particular construct; in this case the construct was an algebraic group.
- The second column listed the number of class meetings that had occurred since the formal definition of a group had been given that the example at hand demonstrated. A written homework assignment was coded as occurring on the day that it was assigned. This was meant to help describe its approximate position in the example neighborhood.
- The third column described the qualities of the example or non-example. In the case of examples, the third column described any additional qualities that the example possessed from a list that would be known to first semester algebra students by the midpoint of the semester (such as being a commutative group, a finite group, or a cyclic group). For non-examples, we described any properties of the construct that were missing as well as additional properties that the non-example possessed from the list above. This was meant to allow us to describe the range of permissible variation that was included in the enacted example space.
- The fourth column described the manner in which the example or non-example was made part of the classroom discourse as well as how the example was used. This was meant as a way to describe the example function.

Subsequently, we summarized the example space and the range of variability that was part of the enacted curriculum of the class as described in the section on our theory of the enacted example space. To describe the example neighborhood we present a narrative that describes each of the examples and when, in time, they were presented. We pay special attention to the first example and what other examples are presented soon after. We also described how the examples varied from one to the next, and, following Dienes (1963) suggestion, whether those examples varied along more than one dimension at a time. Finally, we give a description of the total variation in the qualities of the examples and non-examples that the students experienced. Collectively, we believe that these analyses allow us to describe the enacted example neighborhood.

After describing the example neighborhood, we analyzed all of the constructed examples. We first described the examples that were constructed by drawing on column four in the table. We also described the methods for constructing examples that the students was exposed to. The goal of describing the construction methods is to be able to predict the tools that students have to create their own examples, including novel ones.

In the third phase of the analysis, we drew on the fourth column of the table to describe the function of the example. The great majority of examples were used to exemplify a particular concept. There were relatively few examples that were used for any other function, but, for example, we did capture an instance of using an example to understand a theorem. We did this by describing the context of each example and how, if at all, the students or instructor made subsequent use of the example.

The fourth column also provided enough information to describe how the students engaged with the material, if at all, by describing how it became part of the classroom discussion and how it was used. From this column we described how students engaged with an example; for example, they might have all worked on the example during class. The example might have been assigned as part of a homework assignment to be submitted and, therefore, all students would have engaged with it in some form. Or the example might have been part of classroom discussion but without all students having been required to work on it. Finally, by drawing on the way that students engaged with examples, as well as all of the previous analyses, we drew inferences about the students' ability to learn and their likelihood of learning. In the case study presented below we will first present the collection of examples that the students experienced. Then, we will present subsections describing each of the aspects of the example space.

Data and Analysis

In the analysis that follows we present and analyze data immediately following the instructor's introduction of the formal definition of a group. These class periods drew on more examples per day than any other. They also featured a wider range of example and uses of example than others. We feel that the class periods directly following the introduction of the definition of a group are the best illustration of the instructor's teaching with examples.

The instructor introduced the definition of a by discussing the historical roots of algebra as a study of solving equations and asked the question, "In the case of equations of the form $a*x=b$, what properties do we require of the set and operation to be able to solve the equation?" He presented $x+3=5$ and $a*x=b$ and then drew out the required properties for the definition of a group.

4.1 Dr. P's Teaching

In the lecture-based course the instructor, Dr. P, defined a group and showed 6 structures within the same class period. Four of the structures were examples of groups. The first example was (Z, +), immediately followed by (Q, +). Both were asserted without proof based on previous work. He next proposed the rational numbers under multiplication as an example of a group, but immediately noted that 0 does not have an inverse. As a result, he stated that the structure is not a group. He then proposed a modification of the set to be the non-zero rational numbers and then verbally checked each of the required properties.

Next, he claimed that the non-zero real numbers under multiplication are a group for similar reasons. Dr. P. next proposed a set and arbitrary operation $(R-\{-1\},*)$ and asked, "How can we define * in such a way that this is a group?" He proceeded to define $a*b=a+b+(ab)$ and gave a verbal check that all properties held. Finally, he introduced the group $(Z_{12},+_{12})$ as an example. The entire sequence of discussion lasted approximately 25 minutes. The only other example introduced over the next two class periods was the more general case of the integers modulo *n*, $(Z_n,+_n)$. On the third day after introducing the definition of a group, Dr. P gave examples of the integers modulo 6 and 12 under appropriate additions and then spent 35 minutes proving that $(Z_n,+_n)$ is a group. He then introduced the concept of cancellation in groups via an example in the integers, $3+x=5$. He proceeded to solve the equation for *x* by substituting 3+2 for 5 and writing $x=2$. He asked, "Can we prove that we can always cancel by using our group axioms?"

On the fourth day after the definition of a group he asserted that 'mixed' cancellation ($a*b=c*a$ implies $b=c$) is not a logically valid claim and then stated, "we need some non-commutative group examples," but did not supply any. After asserting this, he asked, "when do groups have the same structure?" and drew on the examples of $(Z,+)$, $(Q,+)$, $(Z_{12},+_{12})$, and $(E,+)$ where *E* is the even integers. Dr. P gave a verbal explanation as to why $(E,+)$ is a group. He repeated his question, "when can these groups be the same?" He continued by saying, "We're in the driver's seat. We get to define this" (meaning, the sameness of groups). Dr. P then asked the class to vote on which of the examples they felt should be, "isomorphic" (he used the term without definition) or the same. The results were such that a plurality want $(Z,+)\cong(Q,+)$ and $(Z,+)\cong(E,+)$ while no students voted for any equivalence with the finite group or between $(Q,+)$ and $(E,+)$. In this case, the students showed some evidence of engaging with the examples.

On the fifth day after the definition, Dr. P drew on two different presentations of a group of 2 elements and showed that they were isomorphic to each other and to $(Z_2,+_2)$ both by rearrangement of the operations tables and renaming of the operation and then by using the formal definition of isomorphism of groups. Dr. P then showed that $(Z,+)\cong(E,+)$ and $(Z,+)$ is not isomorphic to $(Q,+)$. Finally, Dr. P assigned a set of practice problems that included a number of examples and non-examples.. Eight of the problems required students to determine if a set and operation formed a group or a subgroup of a specified group. Of those eight, five did not satisfy the properties of the group definition, with three of the non-examples being commutative and two non-commutative. The other three items included two commutative groups and one non-commutative group (the upper-triangular subgroup of the general linear group of *n* x *n* matrices with real number entries). The students were also to exhibit an abelian group of 1000 elements and to show that the complex numbers with a norm of 1 are not isomorphic to either the real numbers under addition or the non-zero real numbers under multiplication.

On the 7th day after the definition, Dr. P introduced the symmetric group on 3 elements and gave out the second required homework set. As part of that problem set, the students were to exhibit a group, *G*, which has a non-abelian subgroup, *H*, and then to justify their response. The students were highly likely to engage with this problem because it was homework to be submitted. Finally, on the 8th day, Dr. P spent the class proving the claim that every finite group of even order has an element of order two. He illustrated this claim with the examples of $(Z_{12},+_{12})$ and $(S_3,\circ)$.

Day 0	(Z, +), (Q, +),	
	$(Q,\cdot)$	Not a group
	$(Q^*,\cdot)$, $Q^* = \{q \in Q \mid q \neq 0\}$	Constructed
	$(R^*,\cdot)$	
	$(R-\{-1\},*)$	Constructed * such that the result is a group.
	$(Z_{12},+_{12})$	Introduced, not formally defined.
Day 3	$(Z_6,+_6)$, $(Z_{12},+_{12})$, $(Z_n,+_n)$	Formally defined, the first illustrated the general case
	$3+x=5$ implies $x=2$ by cancellation	Using an example from the integers to illustrate the idea of cancellation.
Day 4	(Z, +), (Q, +), $(Z_{12},+_{12})$, and $(E,+)$	When do groups have the same structure? When are they the same?
Day 5	3 different representations of $(Z_2,+_2)$	Showed they were isomorphic by re-arranging operation tables and via the formal definition
	Showed $(Z,+)\cong(E,+)$ $(Z,+)$ is not isomorphic to $(Q,+)$	Illustrating formal definition by showing that $(Q,+)$ is not cyclic.
	five did not satisfy the properties of the group definition, with three of the non-examples being commutative and two non-commutative. All infinite. The other three items included two commutative groups and one non-commutative group. All infinite exhibit an abelian group of 1000 elements show that the complex numbers with a norm of 1 are not isomorphic to the real numbers under addition or the non-zero real numbers under multiplication. Infinite and commutative	Practice problems
Day 7	$(S_3,\circ)$	Introduced and defined
	exhibit a group, *G*, which has a non-abelian subgroup, *H*,	As part of homework
Day 8	$(Z_{12},+_{12})$ and $(S_3,\circ)$.	Illustrating a claim that every finite group of even order has an element of order 2

4.1.1 Dr. P's enacted example neighborhood

The first example of a group that the instructor gives is the group of the integers under addition, an infinite cyclic group. The next three structures are all infinite and commutative, with the only variation the students experience being a change in either set or operation. The third proposed structure was a non-example, as the rational numbers under multiplication do not form a group, but was infinite and commutative. Thus, the students had early access to the idea that not all common sets and operations form a group. The fourth structure was a modification of the third, and, included the first new set, the non-zero rational numbers. The next two examples were similarly common sets, each missing one element, along with a commutative operation. The sixth operation was non-standard and offered students the opportunity to expand their example space by expanding their range of operations. The final example introduced on the same day as the definition was a finite, cyclic example, this was the first finite example that the students had experienced. By the third day after the introduction of the definition of a group Dr. P introduced a way to create finite cyclic examples of any size. On the fourth day after the introduction of the definition, Dr. P introduced a new example, the even integers under addition, as a way to illustrate the concept of isomorphic groups. This group is also an infinite and cyclic group with a known operation and reasonably common set. As a result, while the group is new to the students, it may not expand their example neighborhood in any significant way.

Thus, by the end of the fourth day, the students may have an example neighborhood that was populated by examples of groups of all possible sizes, that includes at least one non-standard operation, and included non-standard sets such as all of the real numbers except negative 1. Moreover, the students have access to the fact that not all sets and operations form groups. What the students have not yet experienced is a non-commutative example of a group, nor a non-commutative structure of any type. Some limitations to this enacted example space are evident: all the groups given were commutative, and only one non-example was proposed (and it was quickly revised into an example). It is possible that the students could develop significant misconceptions about the nature of groups, including the fact that all groups are commutative. Thus, we believe that this instructor did not offer his students a mathematically rich example neighborhood in the first four class periods. Furthermore, the non-example offered was insufficient to allow the students the opportunity to understand the limits of the concept of group. By this time Dr. P had started using the examples to illustrate concepts as well as propositions and had not given students an intellectual need to include non-commutative examples. We believe that this makes it likely that students would mistakenly believe that all groups are commutative. Similarly, because the students only saw one non-example and it was modified into an example, they may believe that all non-examples can be made into examples using the same modification strategy, or another one. Finally, we note that the students were expected to engage with the examples of $(Z, +)$, $(Q, +)$, $(Z_{12}, +_{12})$, and $(E, +)$ when they were asked to vote on which of them were isomorphic.

In the fifth class period after the introduction of the definition of a group the students experienced an additional variation in their example space. In that class period Dr. P assigned practice problems that included working with matrices, meaning that students were introduced to non-commutative structures. The first three non-commutative structures were also not groups, while only one of the non-commutative structures was a group. In order to complete all of the problems the students would also have needed to develop an understanding of the general linear group, another non-commutative group. All of the examples were infinite, and, just as Dienes (1963) suggested, they varied by one dimension at a time. On the seventh day after the introduction of the definition of a group Dr. P presented the permutation group on three elements. This was the first finite non-commutative group that the students experienced. Moreover, they were required to engage with such a group in the homework that Dr. P assigned that period.

The summary of the variation that students experienced is as follows: they saw commutative groups of all sizes, an infinite non-commutative group, and a 6 element non-commutative group. They saw cyclic groups of all sizes and infinite non-cyclic groups, including a proof that the rational numbers are not cyclic. The students experienced some artificial constraints on the variation of their example space; specifically relating to the sizes of non-commutative groups. Moreover, because the first three examples of non-commutative structures that students experienced, and, no non-commutative structures were introduced until five days after the definition of a group, we hypothesize that students are likely to hold the mistaken belief that all groups are commutative. In terms of how the variation was presented, with the exception of the introduction of the permutation group, it followed Dienes' (1963) recommendation about minimizing the amount of variation from one example to the next. Thus, we believe that the students had the opportunity to construct a well-defined example neighborhood, but, it was generally populated and defined by familiar structures.

4.1.2 Example construction

The instructor offered two different structures that were made into examples of the concept of group. The first was proposed, shown to not satisfy the axioms, and modified by removing the problematic element from the set. This gave the students the opportunity to learn one possible example construction strategy: remove problematic elements.

The instructor proposed another structure and asked what definition of an operation would make the structure a group. This called into being a non-standard operation. It gave the students access to a new range of operations as well as a new way to construct examples. They might have learned that they could begin with a set and then define an operation, standard or non-standard, that would make the structure a group. The instructor did not give the students a meaningful opportunity to learn how he created the operation, however, so it is not clear that the students would be able to immediately adopt the second example construction process. Given that the instructor offered two different methods, both illustrated with exemplars of the process, to construct new examples of groups, we believe that his instruction was of likely to help the students develop methods for constructing examples.

In their homework, the students were expected to exhibit a group with a non-commutative subgroup. Students may have attempted to construct another example using one of the two illustrated methods. But, they could have done this without constructing a new example by saying that a group is a subgroup of itself and then exhibiting $(S_3,\circ)$.

4.1.3 The function of examples

There were 4 different functions of examples illustrated in Dr. P's teaching. He used examples to illustrate definitions such as a mathematical group, isomorphic group, and an isomorphism of groups. He also used examples to instantiate statements of propositions after they had been introduced, such as when he drew on examples while discussing the claim that every finite group of even order has an element of order two. Dr. P used examples as a means to motivate the need for and introduce definitions of new constructs. For example, before he gave the definition of group or isomorphism he introduced the idea via a concrete example such as, "what properties of the set and operation are necessary in order to be able to solve this equation?" Finally, Dr. P used examples to motivate claims (he was not observed generalizing from examples, thus, we have chose the phrase motivate rather than create) before giving their formal statement such as when he drew on solving equations in the integers to motivate the question of whether cancellation is possible in groups.

With respect to the second aspect of the function of example, $(Z_{12},+_{12})$ was the most frequently cited example. It was used as an exemplar of two different concepts, a group, and when two groups are isomorphic. The group was also used to illustrate a claim about finite groups. Thus, we claim that in the enacted example space, $(Z_{12},+_{12})$, will possibly start to occupy a preeminent status. Given that it is the most frequently invoked, it would not be unreasonable for student to link example of $(Z_{12},+_{12})$ with the definition of a group as their concept image. This would be problematic due to the fact that $(Z_{12},+_{12})$ as a concept image would not allow students to reconstruct the definition of a group as it has additional properties.

Significance and Directions for Future Study

This paper makes four meaningful contributions to the research literature. The first significant aspect of this study is that it describes and analyzes the teaching of an undergraduate abstract algebra class. Before this, there were relatively few studies of undergraduate instruction in proof-based courses and none of an abstract algebra class. As this is a single case study, it is inappropriate to draw generalizations from it; yet, without a body of empirical evidence there is no basis for more theoretical work.

The second contribution of this study was to provide a finer-grained description of the example space. Mason and Watson (2008) described many of the criteria outlined in this paper, including knowing what can vary and what must stay constant, but we have added detail in the areas of example construction, such as being specific about what techniques for example construction students have access to. Similarly, we have added detail about function of examples, where the uses examples may be described. We believe that the construct of example function will give researchers a tool to analyze teaching as well as a way to better understand students' personal example neighborhoods. It will also provide more detail about students' concept image (Vinner, 1991) and their accessible example space.

The third contribution of this paper was a more detailed description of the example space as a tool for studying undergraduate teaching. Generally, "researchers' questions, methods, and analyses have not generally targeted what teachers say, do, and think about collegiate classrooms in an extensive or detailed way" (Speer et al., 2010, p. 105). These studies have failed to examine the teaching of any content in undergraduate classrooms in depth. In addition, there is an insufficient quantity of tools for doing so (Rasmussen and Marrongelle, 2006). This study created a new lens for analyzing undergraduate proof-based classes that gives insight into what students might learn. Although this technique was demonstrated in the setting of abstract algebra, we assert that it would be equally meaningful in any other proof-based undergraduate course.

This analysis provides a direction for future research, in addition to analyzing the enacted example functions. In particular, we propose to connect the analysis of the enacted example space with what students actually learn. This paper specifically described the opportunities students had in an algebra class to learn about examples of groups. We propose to investigate this question: does an increase in the number of examples discussed in class actually translate into students having richer individual example spaces?

Furthermore, while there is theoretical work describing the importance of students' example spaces and personal reflection from mathematicians that a rich example space helps in their work, there is no evidence that students at the advanced undergraduate level with a richer example space are more able to abstract, generalize or write proofs. We propose to investigate this correlation as well. We believe that generally a richer example space is better, but that the level of comfort and fluency with examples is also important. Neither of these concerns are adequately addressed in the analysis above.

Finally, we propose that this method of analysis gives undergraduate mathematics faculty and those who work with them a tool that may help reform instruction. More careful thought about the way that examples are presented, sequenced, and the functions that they are put to may cause faculty to think practice better pedagogy. Again, we present this as a possibility that would need further work in order to develop it.

References

Alcock, L. (2009). Mathematicians' perspectives on the teaching and learning of proof. *CBMS Issues in Mathematics Education, 16*, 73-100.

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111-129.

Bills, L. & Watson, A. (2008) Editorial introduction. Special Issue: Role and use of exemplification in mathematics education *Educational Studies in Mathematics*. 69: 77-79

Courant, R. (1981). Reminiscences from Hilbert's Gottingen. *Mathematical Intelligencer*, *3*(4), 154–164.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics, 33*, 283–299. doi:10.1023/A:1002999415887.

Dienes, Z. (1963). *An experimental study of mathematics-learning*. London: Hutchinson Educational.

Goldenberg, P. & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics, 69*, 183–194. DOI 10.1007/s10649-008-9143-3

Harel, G., & Fuller, E. (2009). Contributions toward perspectives on learning and teaching proof. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 355-370). New York, NY: Routledge.

Harel, G., & Sowder, L. (2007). Towards a comprehensive perspective on proof. In F. Lester (Ed.), *Second handbook of research on mathematical teaching and learning* (pp. 805-842). Washington, DC: NCTM.

Lakatos (1976). Proofs and Refutations. Cambridge: Cambridge University Press.

Larsen, S. and Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics, 67,* 205-216.

Mason, J. & Watson, A. (2008). Mathematics as a Constructive Activity: exploiting dimensions of possible variation. In M. Carlson & C. Rasmussen (Eds.) *Making the Connection: Research and Practice in Undergraduate Mathematics.* (pp189-202) Washington: MAA.

Mejia-Ramos, J. P., and Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 88-93), Taipei, Taiwan.

Michener, E. (1978). Understanding Understanding Mathematics. *Cognitive Science*, 2 361-383.

Rasmussen, C. & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics into instruction. *Journal for Research in Mathematics Education*, *37*, 388-420.

Rasmussen, C. & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly & R. Lesh (Eds.). *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (Pp 195-215). Mawah, NJ: Erlbaum.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44–55.

Speer, N., Smith, J., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*, 99–114.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordecht: Kluwer.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: learners generating examples*. Mahwah: Erlbaum.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior, 23*, 115-133.

Weber, K. (2010, February). *The pedagogical practice of mathematicians: Proof presentation in advanced mathematics courses*. Conference on Research in Undergraduate Mathematics Education, Raleigh, NC. Retrieved from http://sigmaa.maa.org/rume/crume2010/Archive/Weber_Keith.doc.

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics, 69*(2), 131-148.

Zodik, I. & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*, 165–182 DOI 10.1007/s10649-008-9140-6.