Concepts Fundamental to an Applicable Understanding of Calculus

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Calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their discipline courses. Unfortunately, many students leave calculus with an exceptionally primitive understanding and are ill-prepared for discipline courses. This study has begun to work with presumed experts (undergraduate mathematics and other discipline faculty members) to develop a small number of prototype tasks that will elicit, document, and measure students' understanding of a few calculus concepts the faculty participants believe to be essential to successful academic pursuits within their respective disciplines. This paper discusses the data and analysis from the first round of interviews. Implications of these findings for calculus curriculum are presented.

Keywords: Calculus, understanding, model-eliciting activities, design research

According to Ganter and Barker (2004):

> *Mathematics can and should play an important role in the education of undergraduate students. In fact, few educators would dispute that students who can think mathematically and reason through problems are better able to face the challenges of careers in other disciplines—including those in non-scientific areas. Add to these skills the appropriate use of technology, the ability to model complex situations, and an understanding and appreciation of the specific mathematics appropriate to their chosen fields, and students are then equipped with powerful tools for the future.*
>
> *Unfortunately, many mathematics courses are not successful in achieving these goals. Students do not see the connections between mathematics and their chosen disciplines; instead, they leave mathematics courses with a set of skills that they are unable to apply in non-routine settings and whose importance to their future careers is not appreciated. Indeed, the mathematics many students are taught often is not the most relevant to their chosen fields. For these reasons, faculty members outside mathematics often perceive the mathematics community as uninterested in the needs of non-mathematics majors, especially those in introductory courses.*
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> *The mathematics community ignores this situation at its own peril since approximately 95% of the students in first-year mathematics courses go on to major in other disciplines. The challenge, therefore, is to provide mathematical experiences that are true to the spirit of mathematics yet also relevant to students' futures in other fields. The question then is not whether they need mathematics, but what mathematics is needed and in what context. (p. 1)*

These claims of Ganter and Barker detail the rationale for The Mathematical Association of America's (MAA) Curriculum Foundations Project (CFP, http://www.maa.org/cupm/crafty/cf_project.html). The CFP studied the first two years of undergraduate mathematics curriculum. Portions of the mathematics community and its partners, what I refer to as "client" disciplines (e.g., biology, business, chemistry, computer science, several areas of engineering), worked together to generate a set of recommendations that have assisted mathematics departments plan their programs to better serve the needs of its client disciplines (Ferrini-Mundy & Gücler, 2009).

The push to better serve the needs of client disciplines stemmed from the calculus reform efforts. Between the mid 1980s and the early 1990s, the undergraduate mathematics community engaged in a concentrated effort to overhaul the teaching and curriculum of beginning calculus (Ferrini-Mundy & Gücler, 2009). The heart of the reform was the concern over the depth and breadth of students' understanding of calculus (Douglas, 1986). This lack of understanding became especially apparent when students were asked to apply calculus in unfamiliar situations (Hughes Hallett, 2000), such as those encountered in client courses.

As Ganter and Barker (2004) implied, client department faculty often complain that students are unable to apply calculus in the client coursework. This coursework often asks students to use the calculus concepts in ways not familiar to them. For example, the minimization of average cost is done symbolically in calculus, whereas it is usually done graphically in economics (Lovell, 2004). At other times, even when the concept is used in a similar fashion, differences in notation or a lack of familiar cues (e.g., "maximum" or "minimum" in an optimization problem) derails students. Such difficulties in transferring knowledge between disciplines are stark indicators of a lack of understanding (Hughes Hallett, 2000). Thus, the reform called for fundamental changes in the curriculum and pedagogy of beginning calculus.

Therefore, the purpose of this study is to work with presumed experts to develop a small number of *prototype* tasks to measure understanding of a small number of calculus concepts that all agree are important. Correspondingly, three products will emerge from this study:

1. A determination of the fundamental calculus concepts.
2. An analysis of what it means to understand each concept.
3. A pool of tasks that elicit, document, and measure the type and level of understanding a student has of these concepts.

Together with the help of the researchers, the presumed experts have begun developing prototype tasks (product 3) that are informing products 1 and 2. This study is ongoing, and as such, the results discussed here represent the first iteration: the initial determination of the fundamental calculus concepts and what it means to understand a concept, as well as examples of potential tasks that elicit understanding.

Study Description and Methodology

The changes that took place during the reform years placed greater emphasis on conceptual understanding (Hughes Hallett, 2000) rather than procedural skills[1] (Ferrini-Mundy & Gücler,

[111] Ferrini-Mundy and Gücler did not define either *conceptual understanding* or *procedural skills*. To establish a common definition, for the purposes of this study, I define "understanding calculus beyond computations" as an intertwining of computational skill and conceptual understanding. It is the relationships between skill and understanding that give a student the power of application in a wide variety of settings. "In order to learn skills so they are remembered, can be applied when they are needed, and can be adjusted to solve new problems, they must be learned with understanding" (Hiebert et al., 1997, p. 6). Power and flexibility in

2009), but as Ganter and Barker (2004) pointed out, it has not been enough and the disconnect between what the client disciplines need and what the calculus courses provide still exists. Not only does the "what mathematics is needed and in what context?" question remain, it must be extended to include understanding and assessment, rather than merely calculus computations and procedures. Following in the footsteps of the MAA's Curriculum Foundations Project, while narrowing the mathematical content focus to calculus, this study began exploring the potential disconnect between the calculus taught in the mathematics classrooms and the calculus needed outside the mathematics classrooms at a particular undergraduate engineering institution. Through exploring the disconnect, this study began to identify some of the fundamental calculus concepts students need for successful academic pursuits outside the calculus classroom, to illustrate what it means to understand a concept, and to draft tasks that elicit, document, and measure student understanding of these concepts.

Describing the fundamental calculus concepts and creating the pool of tasks constitutes a "design research" study (Brown, 1992; Collins, 1992). In design research, the goal is to put people with different perspectives into situations that require them to express not only how they think about a concept, but to express it in a way that requires them to test and revise their way of thinking (Lesh, 2002). As such, each cycle, or interview session, includes divergent ways of thinking, selection criteria for the most useful ways of thinking, and sufficient means of carrying forward the ways of thinking so they may be tested and revised during the next cycle. Diversity, selection, and accumulation are necessary for iterative revisions to be passed forward.

The framework for this study can be thought of as a multi-tier design experiment (Lesh & Kelly, 2000). There are three tiers in this research project: 1) researchers/facilitators, 2) faculty members/researchers, and 3) students. For the type of experiment described here, the goal is not to produce generalizations about students, faculty members, or groups. Instead the primary goal is to work with presumed "experts" (instructors that have taught a course of interest two or more times) to develop a small number of prototype tasks that measure calculus understanding. Our role as researchers/facilitators is to initiate, document, and analyze the descriptions of the fundamental calculus concepts, frameworks for understanding these concepts, and associated tasks that were elicited from the faculty participants. The role of the faculty participants is to express, test, and revise their thinking about the concepts, frameworks, and tasks through lens of their teaching experience and student work. The third tier, students, will participate in Cycle 2 and beyond. The students will be asked to express their thinking with respect to particular calculus concepts by working several potential tasks. This (anonymous) work will be discussed with faculty participants to test their thinking on what it means to understand the concepts and how to elicit the appropriate understandings. The focus of this study is on the concepts, frameworks, and tasks, not student work. Student work is the medium with which we will discuss and analyze the needed calculus.

Because first-cycle interpretations are expected to be "rather barren and distorted" (Lesh & Kelly, 2000, p. 206) when compared to those that will emerge from subsequent cycles, a series of modeling cycles is usually necessary. Each cycle should challenge both the faculty members and researchers to explicitly communicate, test and assess, and reflect upon their current interpretation(s). At the end of each cycle, the investigators will have refined, reorganized,

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mathematics occur when understanding and skill develop together and are not compartmentalized or context bound (Bruner, 1973; Tulving, 1983).

extended, or rejected their current interpretation(s). Particularly, for the researchers, data interpretation should not wait until all the data has been collected. Instead, data interpretation and analysis will take place throughout the study.

Select faculty members from an undergraduate engineering institution's mathematics and client discipline departments were recruited to participate in an iterative series of interviews centered on generating tasks that will elicit, document, and measure students' understanding of the calculus concepts the faculty participants believe to be essential to successful academic pursuits within their respective disciplines.

For the first cycle, faculty participants were grouped by course (or discipline, as necessary). These intradisciplinary groups consisted of two to four members plus a researcher. For the remaining cycles, participants will be regrouped into interdisciplinary groups to juxtapose different perspectives on calculus.

The interviews were designed around a series of concept descriptions, frameworks, and tasks developed by the researchers and/or adapted from the research of others. The intention was to provide scaffolding for the faculty to evaluate and recognize not only the necessary calculus concepts, but the ways in which the concepts need to be understood. During further interview session, research-based tasks and participant-authored tasks will serve to provide a means to elicit, document, and measure the understanding students have of these concepts.

Nineteen faculty members participated in the first interview session. Mathematics and client discipline faculty members were recruited based on their teaching experience with certain courses: single-variable differential calculus (Calculus I), integral calculus (Calculus II), and client discipline courses that listed Calculus I and/or Calculus II as a pre- or co-requisite. Of the 19 faculty participants, 6 taught math, 6 taught physics, 3 taught electrical and computer engineering, and 4 taught operations research[2] (3 of the operations research instructors were members of the Mathematics Department and the other was a member of the Management Department). Together, these instructors taught an average of 6.16 years at this particular institution, with a minimum of 2 years and a maximum of 20 years. Note: Every student at this institution, regardless of major, is required to complete an extensive core curriculum, which includes two semesters of calculus and many calculus-based science and engineering courses.

Before presenting the results and discussion for the first round of interviews, one comment must be made regarding the chosen client discipline courses. As mentioned above, the courses were selected because Calculus I and/or Calculus II was listed as a pre- or co-requisite. Surprisingly, the instructors of several of these courses would not consider them to be calculus-intensive or even calculus-based.

- One faculty member declined participation because she felt the astronautical engineering courses of interest to us did not have "sufficient material to be of use" to this study because "at no point in the semester are students actually required to perform calculus operations, [although] we talk about derivatives." For this faculty member, using calculus concepts appeared to be synonymous with computation and not with discussions or applications.
- One physics faculty participant described the delicate balance within the core physics courses: "If we go out of our way to chase all of the calculus, what we'd end up doing is

[2] Operations Research is the application of quantitative techniques to managerial decision-making.

flunking half the course; at which point we'd get a call down from the Dean. What's effectively happening is that [the incorporation of calculus is] ramping. So little by little each year, it gets a little bit tougher. So the whole class is ramping up. So in 10 years, we'll be able to say [the physics core course is] a calculus-intensive course. But certainly there's more calculus in it now than there was last year and more last year than the year before."

- The operations research faculty participants admitted that calculus is a pre-requisite "for mathematical maturity more than just the actual calculus" and because "the way [the course] is taught, you can do it without calculus."

These comments and others influence how many of the participants view calculus and the required understanding.

Results and Analysis

The first round of interviews began with a very general discussion of calculus and understanding and then progressed to the very specific. Each interview culminated in the faculty participants developing tasks to elicit the calculus understanding they discussed.

Describe Calculus.

To begin the conversation, this description of differential calculus was offered: "Linear functions are easy, non-linear functions are hard, so as much as possible, we approximate non-linear with linear." Following this, the faculty participants offered a barrage of ways to describe calculus. Calculus is …

- Combination of slopes and differences
- Study of change
- Study of infinitesimally small
- Sum of a bunch of little changes
- Tool to solve (complex) problems
- Limit as things get small
- Study of limits
- Use of limiting processes
- Subset of Linear Algebra
- Tool to explain how things work
- Math of instantaneous and continuous change
- Way to deal with a continuum world through discrete and finite concepts
- Adding up small bits of things and understanding how that sums to a bigger thing
- Language to describe what's going on in the physical world; algebra is the grammar

Despite starting from many different ideas, every group eventually described calculus as a *tool*. It is a tool to do something they needed done (e.g., explain how a physical situation works, make a prediction about some physical situation, or solve a problem).

What do you Consider to be the Fundamental Calculus Concepts?

This question was asked twice, once in general and again from the viewpoint of the participants' course(s). Like the descriptions of calculus, the list of calculus concepts began broad. The fundamental calculus concepts are …

- Function
- Infinitesimals
- Infinity
- Limit
- Derivative
- Rate of Change
- Integral
- Summation
- Accumulation
- Riemann sums
- Sequences and series
- Differential equations
- Slope
- Related rates and simultaneously changing rates

As the discussion continued, the participants narrowed this list to the “big” four: function, limit, derivative, and integral. When the discussion shifted focus to the calculus concepts used within client discipline classrooms, everyone agreed on three of these concepts: function, derivative, and integral. The disciplines differed on their view of the limit concept. For physics and operations research, the limit concept “washed out” from the list because other than “the definition of a derivative, you wouldn’t necessarily worry about [limits]” and “we simple don’t talk about limit in any of the courses.” However, both the engineering and mathematics faculty participants highlighted the limit concept. For one math faculty participant, the limit concept is the most important of differential calculus: “the things that I want my students to take away from calculus are the behavior of functions in general and limits. I would be overjoyed if they really had a good understanding of limits. As much as anything else, I think that would be maybe the biggest thing from Calculus I.”

Even though the client faculty participants listed derivative and integral as concepts they used regularly in their courses, they use different verbiage. The users of calculus deliberately use “rate of change,” “summation,” and “accumulation” instead of “derivative” and “integral.” This choice helps them stress the physical meaning of these concepts to the students. For example, one physics faculty participant incorporates meaning and symbols by writing “sum” over every integral symbol on the board. He believes this helps the students understand that the funny “S” symbol means summation. When asked if the physics and electrical and computer engineering participants use this meaning-reinforcing verbiage in industry, they answered no. When working in industry, they use the words “derivative” and “integral” because they can safely assume their peers have the necessary understanding. But with students, they cannot make this same assumption. So, reinforcement of meaning through verbiage (and notation) is necessary in the classroom and they choose to do so with the words “rate of change,” “summation,” and “accumulation.”

One notion that needs to be further investigated is the use of “function” by the faculty participants. Although all the participants used the term “function,” we suspect the concept behind the word is equally different across groups even when the verbiage does not reflect any difference. It could be a language issue, not a content issue.

What Does it Mean to Understand a Concept? How do you Elicit This Understanding?

The assumed understanding the participants spoke of became the focus of the interview sessions, specifically a description of the calculus understanding a student needs to succeed. How each faculty participant group characterized understanding varied depending on their role as a *teacher* of calculus or as a *user* of calculus.

According to *teachers* of calculus, understanding is about the *HOW* and *WHY*. At the end of each calculus course, teachers of calculus want students to walk away with a toolbox full of tools, or procedures, the students know *how* to use, as well as *why* they should use them. Teachers of calculus want students to develop procedural fluency (i.e., to be able to carry out pre-fabricated procedures flexibly, accurately, efficiently, and appropriately). They also want students to develop procedural application (i.e., a student is able to discuss the pros and cons of a procedure, what is needed to apply one procedure versus another, and what procedure is appropriate). They believe “learning skills leads to building concepts.”

How do teachers of calculus elicit the *how* and *why*? For the teachers of calculus, understanding is elicited via written tasks or oral questions. One example of such a task is given in Figure 1. This task walks a student through all the required steps to ultimately graph the given function without the aid of technology. The faculty author of this task believes very strongly that if a student can accurately produce the graph, then that student understands the derivative concept.

> Given a function: $y=2x^4+4\sin 3x,\ -3\leq x\leq 3$.
> Without using technology:
> a . Find the derivative.
> b . Find the minimum(s) and maximum(s).
> c . Find the second derivative.
> d . Find the inflection points.
> e . Where is the function increasing? Decreasing?
> f . Where is the function convex? Concave?
> g . On the following coordinate axis, graph the points from (b) and (d).
> h . Using (a), (c), (e), and (f), graph the function *y*.

***Figure 1*. Task written to elicit understanding of the derivative concept.**

The task in Figure 1 is indicative of the other tasks written by the *teacher* participants. These tasks tended to be situated in theoretical or abstract contexts, in which there are no clues given to what the function stands for, if it stands for anything. More often than not, the context is a contrived situation to get at a particular procedure.

According to *users* of calculus, understanding is about the *WHICH* and *WHY*. Within the client courses, the users of calculus want students to have more than just a toolbox full of tools, or procedures: “It’s not so much that [the students] understand how to turn the crank and spit out an answer. Really mastering [calculus] relies on understanding what that integral or what that derivative actually means in the physical world.” Users of calculus want students to develop *relational understanding* (Skemp, 1978/2006; i.e., recognition of the concept being dealt with and relates it to an applicable procedure). They also want students to develop *predictive power* (i.e., a student is able to step out of the mathematics and recognize that in the mathematics, there is a prediction or truth about what is happening in the physical world). Students need to assess physical situations and select the calculus tool (a pre-fabricated concept and associated procedure) that will enable them to make sensible predictions about the situation. For example, “what does the variable in this equation, that the student just constructed, mean? If it doubles, what happens to the real world? Does the student suddenly get a space ship that flies faster than the speed of light?” If so, something must be wrong with the mathematics.

How do users of calculus elicit the *which* and *why*? The users of calculus elicit understanding via written tasks. One example of such a task is given in Figure 2. Knowing that the student participants will not necessarily have any previous physics or engineering experiences, the users of calculus added some discipline-specific information to the task. Thus, the first part of this task describes an electrical field and point charge so the students are not stymied before they even start.

The electrical field of a <u>point</u> charge is given by $\vec{E}=\frac{kq}{r^2}\hat{r}$, where *k* is a proportionality constant, *q* is a charge, and $\hat{r}$ is a vector from the particle to the point under consideration. Electric fields sum vectorially (superposition), so $\vec{E}_{total}=\vec{E}_1+\vec{E}_2$ for two particles. Using this concept, what is the electrical field at some distance *y* above the center of a horizontal bar of length *L* holding total charge *Q* evenly distributed along its length?

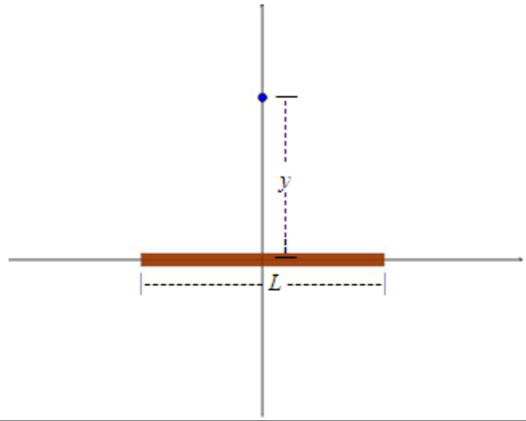


Figure 2. **Task written to elicit understanding of the accumulation concept.**

To successfully complete the task in Figure 2, students need to make a prediction about how much electric field has accumulated at the given point, which requires an integral. This task is situated in the physical world; but, still, the situation is contrived. It fits into a nice, neat little box: an *evenly* distributed charge and the point of accumulation *centered* above the bar. Even with the physical situation, the context is contrived to get at a particular concept and associated prediction.

Similarities in how both the teachers and users of calculus view understanding enable one summary to be put forth. The following presentation of how the faculty participants view understanding may be an oversimplification, it is not as cut-and-dried as it may sound. Nevertheless, the diagram is a useful characterization. According to both teachers and users of calculus, understanding is assessing the given situation and intelligently selecting an existing description (i.e., model[3]) of the expected concept and applying the associated procedures

[3] The definition of "description" used in this paper is basically synonymous with Lesh and Doerr's (2003) definition of model: "*Models* are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently" (p. 10; emphasis original). Lesh and Doerr further characterize a *mathematical* model as a model that "focuses on structural characteristics (rather than, for example, physical or musical characteristics) of the relevant systems" (p. 10).

correctly to get a reasonable answer and/or prediction. Figure 3 diagrams the process and the bulleted statements describe each corner of the 4-part diagram.

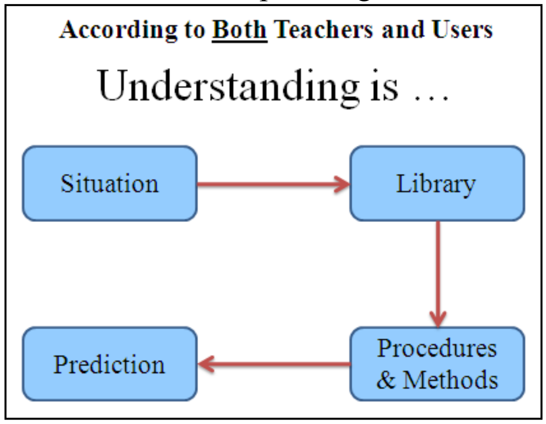


Figure 3. **4-part diagram representing both teachers' and users' views of understanding.**

- **Situation.** Whether tasks are set in a physical, theoretical, or abstract situation, the situation is massaged or formatted such that it fits into an existing description. The students' role is to recognize the expected concept.
- **Library.** Once a student has assessed the situation, he/she must intelligently select the appropriate description from those that already exist in the "library." The teacher's role is to help the students become proficient at recognizing what to "check out" from the library. The students are expected to be able to explain their check-out decision.
- **Procedures and Methods.** Procedures are linked to the choice of description(s) and the student is expected to accurately execute the appropriate procedure.
- **Predictions.** Based on the procedure's result, the student is to make a sensible prediction about the given situation. This includes asking if it makes sense. For example, did the units work out? Would changing a variable yield the expected result?

The teachers and users of calculus are not giving a situation to students and asking them to use calculus to describe and explain that situation. They are simply using contrived situations to help students become better at choosing a description that is already developed and/or constructed for them and then using the linked procedures accurately. The students are not really looking into the situation and using calculus to describe it; instead they are asked to select a description that works.

Discussion and Conclusion

Throughout the first round of interviews, the following thoughts and comments emerged as noteworthy. "Pattern matching," as some participants called it, allows some students to really master calculus, to really understand what the integral or derivative means in the physical world. But for a good majority of students, pattern-matching does not work. Something is missing. One participant asked the researcher if this study was going to change the way calculus was being taught because, from his viewpoint, many of the students were not getting what they needed to succeed outside the calculus classroom. According to the users of calculus, an A in calculus does not mean the student understood and could apply calculus to the physical world. But, according to the teachers of calculus, earning an A means the student really understands deeply. If the purpose of Calculus I and Calculus II is to prepare students for client discipline courses, something needs to change.

Analyzing the interviews and much research, the researchers were led to believe that if someone is using calculus very sensibly (i.e., with a great deal of understanding), he/she can assess a situation and use calculus concepts to describe it. The calculus concepts are linking to procedures, which are then used correctly to make predictions. Unlike before, the process does not stop here. When the individual gets the result and/or prediction, he/she will know what assumptions were made and what things might put some spin on what can be done with the result. This spin may imply some margin of error, degree of uncertainty, or difference made by making one assumption versus another. Those are things that someone who just checks-out something from the library is never going to understand because they never had to. Figure 4 diagrams this hypothesized process, with the differences emphasized with dashed lines. The bulleted statements describe each piece of the 4-part diagram.

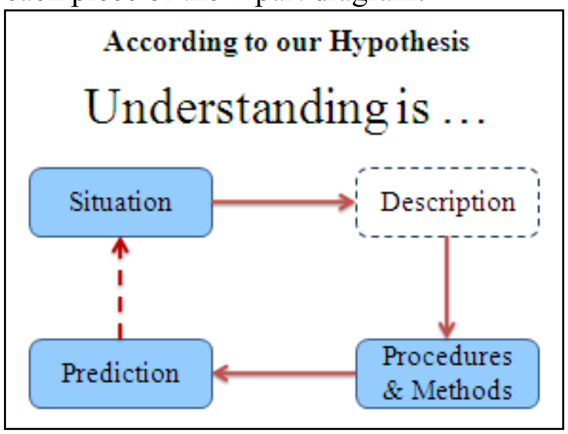


***Figure 4.* 4-part diagram representing our hypothesis view of understanding.**

- **Situation.** Realistic situations are given to students. These situations usually do not quite fit into an existing description.
- **Description.** Students look at a situation, in the raw, and use calculus concepts to compose a description, not necessarily one that matches something in the library.
- **Procedures and Methods.** The concepts used to describe the situation are linked to procedures, which the students are expected to use correctly.
- **Predictions.** Based on the result of the procedure, the students are to make a sensible prediction about the given situation.
- **Relate to Situation** (dashed arrow)**.** The prediction came from making assumptions which spin what can be done with the result. Implies some margin of error, degree of uncertainty, difference made by making one assumption versus another.

Thinking of understanding in this fashion can be likened to cooking. One level of cook will go to the cupboard and assess the ingredient situation. Say, he/she finds tomatoes, mozzarella, and oregano. To this cook, this means an Italian meal and so he/she goes to the library, finds an Italian cookbook and selects a recipe that has only the ingredients in the cupboard. This cook follows the recipe, step-by-step, and produces a meal. It may be an excellent meal, but the cook cannot modify it if needed. If the recipe calls for tomatoes and they are not in season, this cook is forced to find a different recipe because he/she does not know how to modify the recipe. But another level of cook, one with a great deal of understanding, will assess the ingredient situation and compose a recipe instead of choosing one. This person is always going to be a better cook because he/she can take whatever is in the refrigerator, what is fresh and in season, and create an excellent meal. He/she does not have to use tomatoes even when they are not in season.

This second level of cook has the type of flexible, durable, and applicable understanding we want from calculus students. We want a student that can use calculus to describe a realistic situation, one that might not fit any existing descriptions, textbook examples or library entries. When a student arrives at a result, they will know if and how it applies to the given situation and whether it applies to any other situation. This occurs because they know what assumptions they made, what error or uncertainty might be involved. Because the calculus description was created by the individual, he/she owns it. This personal ownership enables him/her to not only build and deepen their understanding, but to use calculus in novel and unfamiliar situations of any kind. This type of understanding is flexible, durable, and transferrable. One faculty participant summed it up this way: "Ultimately we want, not get [students] to be able to do problems similar to what they've seen before, we want them to take what they've known and do something new."

As might be expected, the tasks that elicit this level of understanding look quite different than those presented earlier. Two example tasks are below. The Toy Train Problem in Figure 5 asks students to design a toy train display. They are asked to make the mountain as tall as possible, use the least amount of track, and not exceed a 10% grade on the track. The central calculus concepts students need to use in this problem are rate of change (i.e., derivative) and accumulation (i.e., integral).

The Toy Train Problem

At Christmas time last year, the Indianapolis Children's Museum built a number of large displays using toy trains, gingerbread houses, plaster mountains, and other small toys. Each display covered the top of a large table that was at least 2 meters wide and 3 meters long; and, each display included three or four different sized mountains that were made using paper and plaster. The goals were: (a) to make the mountains as tall as possible, (b) to minimize the amount of train track that is used, and (c) to make sure that the grade of the track never exceeds 10% going up or going down. … But, last year, one huge problem occurred! The toy trains were only powerful enough to go up or down a slope of 10% or less. And, last year, the slopes of most of the tracks were too steep for the trains to run.

The topographic map on the back of this page shows the heights, locations, and shapes of the mountains for one display that was used last year. The table for this particular display was 2 meters by 3 meters; and, the contour lines show the height of the mountains. For example, at the edges of the darkest regions on the map, the height of the mountains is 200 centimeters; at the edges of the next darkest regions on the map, the height of the mountains is 175 centimeters; and so on. … The map also shows the location of the toy village and Santa's Workshop. But, the map does not show the path of the train track that was used. This is because the path of the train track that was used last year was too steep for the trains to run.

<u>Your Task</u>: Write a letter to the people who will be designing the toy train displays this year. Describe how to decide how tall the mountains can be on each table; and also describe if the shapes or locations of some of the mountains need to be changed. To begin, re label the contour lines on the topographic map to show how tall the mountains cane for the display that

is shown on the back of this page. Then, show a path for the train tracks so that the grade will never exceeds 10% going up or down the mountain. Finally, draw diagrams and describe how similar decisions can be made for other tables that have different dimensions. … The display designers need your help as fast as possible. You only have 60 minutes to write your letter and submit your plans.

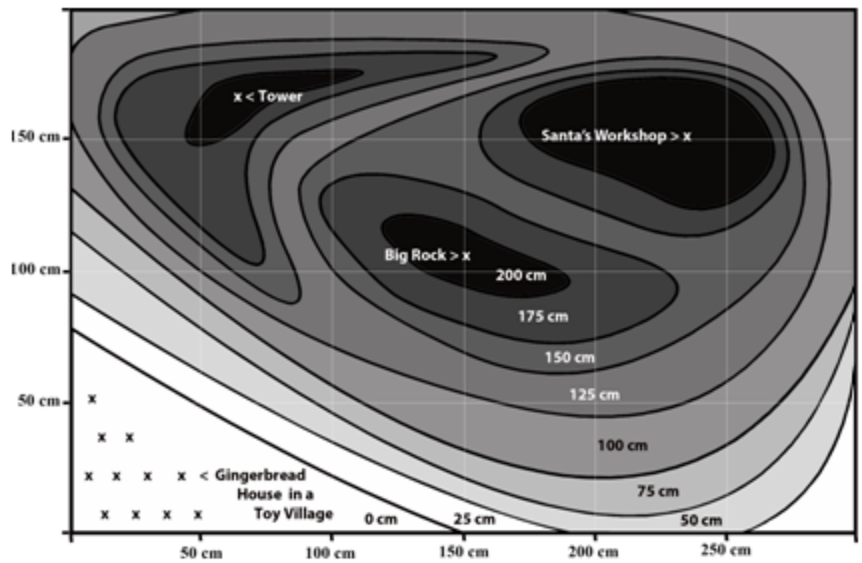


Figure 5. The Toy Train Problem.

The Theater Problem in Figure 6 asks students to create a recommendation for when patrons should arrive at a movie theater to minimize their wait time. Using the given rates and numbers of attendees, the students are asked to minimize the wait time experienced by patrons, while maintaining a 10-minute buffer for "settling in to the movie." The central calculus concepts students need to use in this problem are rate of change (i.e., derivative), accumulation and area (i.e., integral), and the relationship between derivatives and integrals.

The Theater Problem

During last year's summer blockbuster season, the Cinema 12 Theater had horrendous lines at the ticket counter. On several occasions, patrons missed the beginning of a movie or did not have time to buy popcorn and drinks for the show. This lost a ton of revenue for the theater. At other times, patrons waited outside a theater while the cleaning crew finished cleaning up after the previous showing. With the blockbuster season effectively kicking off in a couple of weeks, the theater owner would like to post recommendations on the theater's website regarding how long before a particular showing patrons should arrive. The ideal arrival time should give the patrons time to purchase tickets and leave a 10-minute window for patrons to "settle in" for the movie (e.g., buy popcorn and drinks). The owner is asking for your help in creating these arrival-time recommendations.

The theater owner reviewed last year's records. He found that all the shows between 6 and 9 pm sell out on opening weekend. After that, each showing attracts 75% of each theater's capacity. Each theater has seating for 100 people. Matinee shows (those before 6 pm) attract 40% and late night shows (those after 9 pm) attract 65% of each theater's capacity. To sell tickets, the owner has 2 attendants working at all times, with 2 more standing by, ready to work, depending on the length of the patron line. Each attendant takes no more than 30 seconds to process one patron's request. While the owner wants to minimize the patrons' wait time, he also needs to minimize the number of attendants working.

Your Task: Based on this information, and an example of show times listed below, write a letter to the theater's owner giving your proposed arrival-time recommendations and the number of attendants the owner can expect to work throughout the day. In your letter, describe how you arrived at each recommendation and expectation. Be sure to give details and a clear explanation so the owner can decide whether your recommendations are the ones he should post.

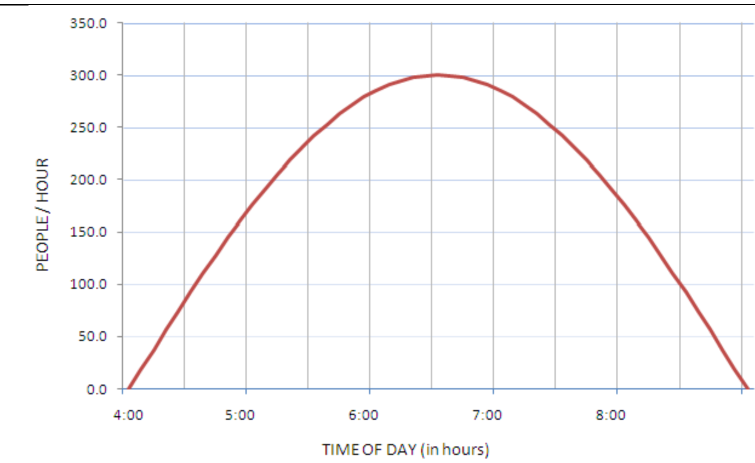
Show times for the Cinemas 12 Theater.

Movie #1	11:10 am	1:50 pm	4:30 pm	7:30 pm	10:25 pm
Movie #2	11:20 am	2:10 pm			
Movie #3	10:45 am	1:30 pm			
Movie #4	10:40 am				
Movie #5	11:50 am	2:30 pm	4:45 pm	7:00 pm	9:30 pm
Movie #6	4:15 pm	7:45 pm	10:30 pm		
Movie #7	10:30 am	1:15 pm	4:00 pm	7:15 pm	10:00 pm
Movie #8	11:30 am	1:20 pm	2:20 pm	4:10 pm	5:10 pm
	7:10 pm	8:00 pm	10:15 pm	10:45 pm	
Movie #9	11:00 am	1:45 pm	4:20 pm	6:50 pm	9:50 pm
Movie #10	4:50 pm	7:40 pm	10:20 pm		
Movie #11	9:40 pm				
Movie #12	11:40 am	2:15 pm	4:40 pm	7:25 pm	9:45 pm
Movie #13	10:50 am	1:40 pm			
Movie #14	10:20 am	1:10 pm	4:05 pm	6:45 pm	
Movie	11:15 am	2:00 pm	4:35 pm	5:00 pm	7:20 pm

	7:50 pm	10:10 pm	10:40 pm		
The times above are example show times (start times for each movie).					

Figure 6. The Theater Problem.

The Theater Problem is a modelling version of a task written by one of the faculty participant groups. The original task (see Figure 7) did not involve varying rates because one of the faculty-author's understanding of cuing theory, which she was currently teaching, did not allow for anything but constant rates. Her faculty partner, an electrical engineer by training and a mathematician through teaching, commented that "it is such a shame to have her say this because having a varying rate is why we need calculus." The levels of calculus understanding demonstrated within this group are what we are trying to elicit, document, and measure with the tasks.



The chart above shows the arrival rate of people coming to a theater box office to buy tickets. The box office has the capacity to sell tickets at a max rate of 200 people/hr.

a . At what time did the line begin to form?
b . At what time was the line the longest?
c . How long is the line when it is at its longest?
d . Is there still a line at 9:00?
e . If you arrive at 6:00, how long will you wait?

***Figure 7*. The theater problem, as written by one faculty participant group.**

Both The Toy Train Problem and The Theater Problem stem from realistic, raw situations that require students to use calculus concepts to describe what is happening and to arrive at a prediction that can be extended to other situations. Note that neither problem specifically requires students to calculate a derivative or integral or even model the situation with a function. This distinction might categorize these tasks, in the minds of faculty members such as the astronautical engineering instructor that declined to participate, as non-calculus tasks. For them, a task that does not require calculus computations is not a meaningful task. These instructors are comfortable with traditional textbook word problems which emphasize computational skills, as evidenced by the tasks written by the participants during the first interview session to elicit calculus understanding.

For those familiar with the Models and Modeling Perspective (Lesh & Doerr, 2003; Lesh, Hamilton, & Kaput, 2007; Lesh, Hoover, & Kelly, 1993; Lesh & Lamon, 1992; Lesh, Landau, & Hamilton, 1983), the tasks in Figures 6 and 7 are very similar to problem-solving activities often referred to as *model-eliciting activities* (*MEAs*). MEAs require students to "go beyond short answers to narrowly specified questions – which involve sharable, manipulatable, modifiable and reusable conceptual tools (e.g., models) for constructing, describing, explaining, manipulating, predicting, or controlling mathematically significant systems" (Lesh & Doerr, 2003, p. 3). MEAs differ from traditional word problems in several ways (see Figure 8). First, in MEAs, students create mathematical descriptions (i.e., models) of meaningful situations (i.e., real world, raw situations). In traditional word problems, students make meaning of symbolically described situations. This difference is the difference between mathematizing and decoding (Lesh & Doerr, 2003). By mathematizing, Lesh and Doerr mean "quantifying, dimensionalizing, coordinatizing, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities" (p. 5). A second difference is what assumptions are made.

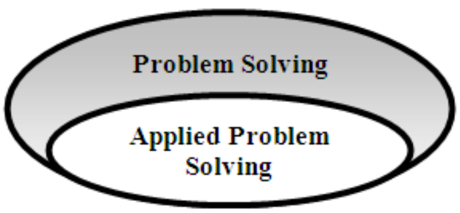
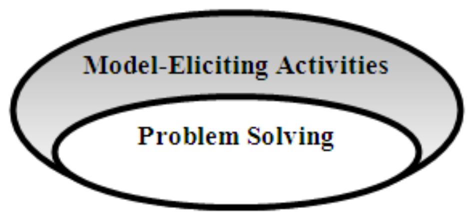
<table>
<tr><td>Traditional Views
Applied problem solving
is treated as a special case
of traditional problem solving.</td><td>Modeling Perspective
Traditional problem solving
is treated as a special case
of model-eliciting activities.</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Learning to solve "real life" problems is assumed to involve three steps:
1 . First, learn the prerequisite ideas and skills in decontextualized situations.
2 . Learn general content independent problem-solving processes & heuristics.
3 . Finally (if time permits), learn to use the preceding ideas, skills, and heuristics I messy "real life" situations where additional information also is required.</td><td>Solving meaningful problems is assumed to be easier than solving those where meaningful interpretation (by paraphrasing, drawing diagrams, and so on) must occur before sensible solution steps can be considered. Understanding is not through of as being as all or nothing situation. Ideas develop; and, for the constructs, processes, and abilities that are needed to solve "real life" problems, most are at intermediate stages of development.</td></tr>
</table>

Figure 8. **Applied problem solving ≠ model-eliciting activities (Lesh & Doerr, 2003, p. 4).**

A secondary purpose of this study is to provide calculus instructors with tasks that can used to measure student understanding of calculus. Once the instructor has this measurement, he/she can adjust his/her instruction to better move the students from where they are to where they need to be. So, in a sense, this study is trying to change the way calculus is currently taught.

This measurement can happen either before or after instruction. If used before instruction, these tasks encourage students to develop their own understandings; and if used after, the student can apply what they have already been taught (Lesh, Yoon, & Zawojewski, 2007). As Yoon, Dreyfus, and Thomas (2010) suggest, if tasks like those described here are implemented after instruction, they not only measure and document a student's understanding, but allow the student to deepen his/her understanding of the calculus concept(s) while affording an opportunity to apply his/her understanding in a realistic situation. Research has shown that for MEAs, students often invent (or significantly extend, refine, or revise) constructs that are more powerful than anybody has dared to try to teach to them using traditional methods (Lesh, et al., 1993).

As stated before, calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their discipline courses. Therefore, at its conclusion, this study hopes to offer a collective vision to focus the content of beginning calculus courses on the meeting the needs of client disciplines. However, in the end, it is the mathematicians that have the responsibility to create courses and curricula that embrace the spirit of this vision while maintaining the intellectual integrity of mathematics. By explicitly knowing what and how students should be prepared for client courses, teachers and curriculum developers of both calculus and client disciplines can work together to prepare students for academic success in any discipline.

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