USING CONCRETE METAPHOR TO ENCAPSULATE ASPECTS OF THE DEFINITION OF SEQUENCE CONVERGENCE

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This study traces how one real analysis student was able to develop a property-based rather than exemplar-based notion of sequence convergence by first instantiating conditions on the parameters of sequence convergence in a metaphorical domain. The analysis uses the radical constructivism framework of cognitive development (von Glasersfeld, 1995). Once he had encapsulated the conditions that related the parameters within the formal definition, he was able to accommodate the conditions into a mathematical schema in which he could reason flexibly about convergent sequences in ways compatible with standard formal practice. I observe his language use and the various lines of reasoning that his concept of the sequence convergence definition did or did not support over the course of a month to identify changes in his convergence schema.

Key words: Real analysis, concrete metaphor, sequence limit definition, radical constructivism

Mathematicians as a community uphold abstraction and generality as hallmarks of mathematical reasoning. The pursuit of abstraction and generality has guided the community toward lines of reasoning built upon definitions and properties rather than examples or categories of examples. The power of formal definitions and deduction thereupon lies in their ability to simultaneously represent entire classes of mathematical objects that satisfy the conditions of the definition. There is broad deductive power in the transition from “This sequence converges” to “For any convergent sequence…” However, research upon the transition from case-based and computational reasoning to definition-based and deductive reasoning (a subset of what has been called Advanced Mathematical Thinking) poses many obstacles for students (Alcock & Simpson, 2002; Tall, 1991). In the realm of calculus and analysis, conceptual aspects of limit have proven quite difficult (Ely, 2010; Oehrtman, 2009) prior to the difficult transition to formal limit reasoning based upon multiple-parameter, multiple-quantifier definitions (e.g. $\forall \varepsilon > 0 \ \exists K \in N$ such that $\forall n \geq K, |a_n - L| < \varepsilon$). The present study investigates how a real analysis professor’s introduction of a concrete metaphor supported one student’s transition toward more formal reasoning both in his understanding and use of the definition of sequence convergence.

Relevant Literature

Three primary pathways have been described for the development of formal definition-based reasoning about definitions of limits: 1) exploring a range of examples to develop a definition that uniquely describes them (Swinyard, 2008), 2) developing "generic examples" through graphical explorations (Pinto & Tall, 2002), and 3) extracting meaning from the formal definition itself (Dubinsky, Elterman, & Gong, 1988). The first pathway via capturing examples holds difficulty in that it takes time and several studies have noted significant differences among the ways different students reason about mathematical definitions (Alcock & Simpson, 2002;

Vinner, 1991) and differences between the ways that students and mathematicians treat definitions (Edwards & Ward, 2008). Alcock and Simpson (2002) distinguished between those students who used examples or sets of examples with which they were familiar to reason about novel examples and those students who used the formal definition to reason about novel examples. Though neither form of reasoning are absent in standard mathematical practice, the mathematical community values the latter because any reasoning depending upon the definition necessarily applies to the entire category of objects satisfying the definition. Edwards and Ward (2008) similarly distinguished between definitions that establish their associated category (stipulated definitions) and definitions that intend to capture a pre-existing category (extracted definitions). In proof-based settings, mathematicians treat definitions as stipulated while students tend to treat definitions as extracted, and this distinction strongly determines the lines of reasoning and conclusions that an individual develops. Vinner (1991) described a similar distinction between students reasoning primarily with their concept image (the set of all ideas, images, examples, and processes associated with a given definition) and students reasoning primarily with the concept definition (the standard set of words used to define a given concept).

Other research has noted that students are capable of developing rich graphical representations for reasoning flexibly about sequences and limits, but in some cases this proves to support development of more formal reasoning in terms of the definition (Pinto & Tall, 2002) and in others it seems to inhibit that development due to the convincing power of the graphical reasoning (Alcock & Simpson, 2004). Finally, in many classrooms in which definitions are presented in final form, students are asked to develop understanding and fluency with the formal definition simply by using it to verify convergence of particular examples or to prove theorems about classes of limits. The present study describes an alternate pathway that one student traversed toward fluency with the standard formal definition using a concrete metaphor for convergence the professor developed during instruction.

Oehrtman (2009) described the metaphors calculus students used to reason about limits contextually and computationally. However, the metaphors he described were not metaphors for the formal definition, but rather metaphors for the notion of limit itself such as numerical approximation or collapsing a dimension. Current work is also exploring how students successfully used sets of examples and the approximation model of limits to develop the formal definition of sequence limits, series limits, and point-wise convergence of a Taylor series (Martin, Oehrtman, Roh, Swinyard, Hart-Webber, current conference proceedings). This study, being informed by Swinyard and Oehrtman's previous work, extends the body of Realistic Mathematics Education (Freudenthal, 1973, 1991) that takes the instructional stance that students should be guided to rediscover mathematics whenever possible. Rediscovery need not follow the historical process by which mathematics was developed, but rather seeks to create pathways by which students can viably develop their own mathematical constructions that approximate standardized algorithms, definitions, theorems, and proofs.

Theoretical Framework

My theoretical framework draws primarily from radical constructivism as articulated by von Glasersfeld (1995). Radical constructivism posits that concepts do not necessarily reflect the objective nature of external reality, but rather are tools that minds use to organize and deal with ongoing experiences. The viability of conceptual structures lies in their ability to account for new experiences and guide the individual’s actions toward desired results, often negotiating problematic situations. In the context of mathematics, individuals construct mathematical conceptions to organize their reasoning and to satisfy the demands of the mathematical tasks in which they are engaged. For a mathematician, these tasks may be self-imposed and be guided by a cultural set of standards for the acceptability of posing and "solving" tasks. For a student, tasks often appear primarily as an element of their educational experience such that tasks are usually introduced by the instructor and standards of "solving" a task derive from the acceptability of their answers in the eyes of their classmates and instructor. In any case, students develop conceptual structures to address the demands of mathematical tasks and assess the viability of those structures either according to an internal sense of coherence (in terms of their pre-existing conceptions) or according to external feedback their mathematical interactions with others elicit.

Once a student has developed conceptual structures for dealing with various types of tasks, there are two primary processes by which new experiences or tasks interact with that structure, which is referred to as a schema. When a student accesses a schema to reason about or guide their action about a novel task or experience, they are *assimilating* the experience into that schema. For instance, a student may have never seen the number 240,831, but they may be able to ascertain many aspects of the quantity because they recognize that this set of numerals can be understood using the place-value number schema. The student may assimilate like this automatically trusting in their line of reasoning until they receive some feedback that their number schema has not led them to a coherent result or they receive input that can no longer be easily assimilated: for instance when they are told that $.999\ldots = 1$. In cases where either a sense of internal consistency, feedback from teachers or classmates upon their own thinking, or new ideas that are presented fail to be viably assimilated by their existing schema (which is referred to as *disequilibrium*), students may alter the schema or reorganize their assessment of the novel input which is called *accommodation*. In the case of $.999\ldots = 1$, the student must engage both processes: first they must identify that their place-value number schema applies to the numbers (assimilation), but they must alter that schema to allow for a single quantity to have multiple representations (accommodation).

The process of organizing sets of exemplars into categories to which a single schema applies has the effect of unitizing or chunking thoughts to ease future processing. Most adults do not spend time reflecting on or adjusting their number schema because it is able to quickly and sufficiently assimilate all of their ongoing experiences with numbers. In this way, people refer to "number" without perturbation, because this abstract set of objects with it's associated rules, properties, and operations can be thought of as a single conceptual structure. Unitizing what at earlier stages of development was a myriad of associated examples, concepts, and processes is called *encapsulation*.

Methods

This study was conducted at a mid-sized university (25,000 students) in the USA. All data in this study was gathered during one 15-week semester of undergraduate real analysis taught by a research mathematician. At this university, this course generally includes a proof-based development of real numbers, sequences, limits of functions, and continuity. At the time of data gathering, the professor had been teaching for over 10 years and had previously taught undergraduate analysis thrice. She has received multiple teaching awards and is widely regarded as an excellent, if not difficult, instructor. The class met twice weekly for 80 minutes at a time and began with 23 students.

Data gathered includes field notes of class meetings, biweekly professor interviews, weekly student interviews with a small group of volunteers from the class, copies of students' written class notes, and copies of interview participant exams. Written field notes recorded all communication written on the blackboards, major aspects and key quotes from professor and student verbal communication, and physical gestures displayed in the discussion of course material.

All interviews were audio-recorded and any written records of the interactions maintained. The student interview participants were selected from among mathematics majors who volunteered so as to represent a variety of final grades in the "Intro to Proofs" course that serves as a pre-requisite to analysis at this university. Thus the 6 interview participants included one or two students each who made an A, a B, and a C in "Intro to Proofs."

The interviews (at various times) invited students to:

- recall and explain definitions, theorems, and proofs,
- explain and assess mathematical statements and proofs,
- recall and explain aspects of classroom discussion,
- relate the nature, content, extent, and quality of their group interactions outside of class,
- complete homework activities or other novel activities,
- explain their reasoning on written exam questions,
- articulate their confusion about any mathematical topics with which they were uncomfortable, and
- comment about their course experiences both positive and negative.

Interviews ascertained students' recall and interpretations of classroom explanations and discussions as well as their strategies and success on questions and activities presented to them.

The rather general nature of the lines of questioning reflects the research intention that interviews reflect and respond to particular events and elements of the classroom dialogue. Interview questions were developed throughout the semester to pursue student understanding and perception of things such as class discussions, particular diagrams, or chains of reasoning that were expressed during class meetings. The interview questioning often moved from very open-ended to very particular so as to initially allow interview participants flexibility in their choices of verbalization and representation. Then, if needed, interview questions centered participants' focus onto particular elements from the classroom presentation that were of interest to the professor or appeared to the researcher central to the fidelity of classroom communication. For example, the interview protocol may begin by asking a student, "What does it mean for a sequence to converge to a point?" Later, students would be asked about particular aspects of the definition or the explanation thereof such as, "What does 'arbitrary but fixed' mean and why did [the professor] discuss it?" or "In the definition of convergence, what are ε, K, and n?" Finally, a student might be asked to construct a proof that a particular sequence converges to a given limit.

Interviews were transcribed and then coded according to the open coding method described by Strauss and Corbin (1998). Categories varied in nature. Some captured broad sets of interactions such as examples of definition use or comments on the structure of mathematics. Other categories organized every appearance of a given question or example during an interview.

Results

Beginning early in the semester, the professor discussed definitions in two distinct contexts: the concrete metaphorical domain and mathematical domain. The class discussed their reasoning within these two domains in terms of three distinct linguistic registers (sets of words associated by meaning or use), which we shall call the metaphorical register, the intuitive register, and the formal-symbolic register. In the context of the definition of sequence convergence, the professor presented the class with an intuitive register definition meant to represent common notions of convergence that said, "A sequence (a_n) converges to L if a_n gets closer and closer to L as n gets larger and larger." However, she pointed out that in the case of $(4 - {}^1/_n)$, five is a possible limit of this sequence by this definition. Figure 1 shows the number line representation the professor used to display this sequence; she commonly used such number line diagrams to discuss sequences. As students proposed different ways to amend the intuitive register definition to be more specific, Zell told the teacher that "if you want to find a party, see where everyone is at." The professor agreed that they needed to find "where everyone is at," and clearly no one is at 5.

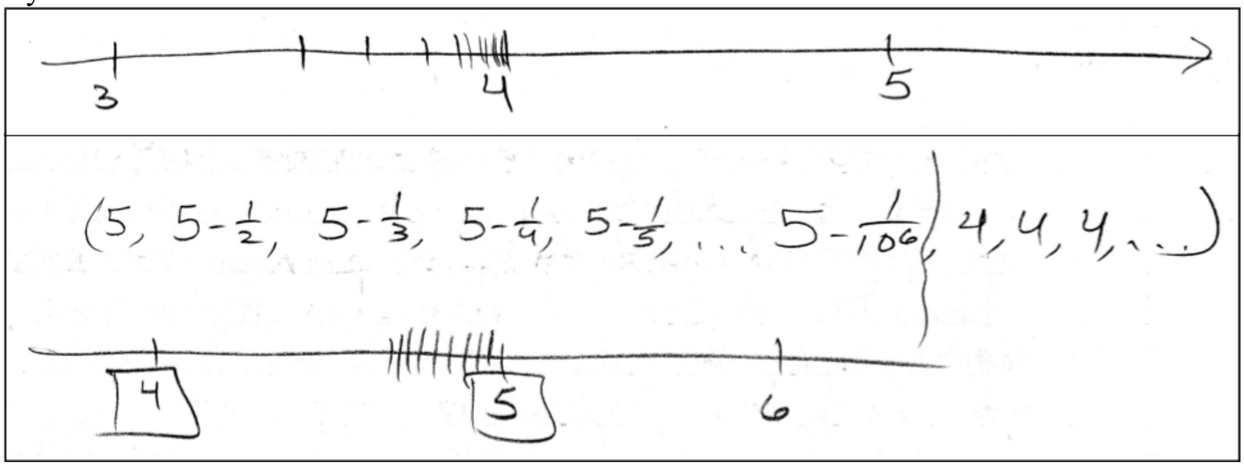


Figure 1. The professor's number line and term-enumeration representations of sequences.

The professor then extended the metaphor asking, "How many terms make a party?" She provided another example that was defined as $5 - {}^1/_n$ for the first million terms and 4 for every latter term. Figure 1 displays the term-enumeration and number line representations that the professor used to display this sequence. She posed the question, "How many people have to be at the party," to which Locke responded "infinitely many." The professor invited the students to talk for a while and share their ideas about how they should form their formal definition. To bring the discussion together again, she wrote on the board a revised intuitive register definition that stated,

"$*_1$ A sequence converges to the real number L if we can make the terms of the sequence stay as close to L as we wish by going far enough out in the sequence."

She verbally mapped this statement into the concrete metaphorical domain saying it is only a party if for any size party you pick, after some point everyone shows up at the party. She then began to ask the class how they could make various aspects of this informal definition "more rigorous." They replaced the previous definition with the following:

"$*_2$ (x_n) converges to L if, given $\varepsilon > 0$, all terms after a certain term x_K are in $(L - \varepsilon, L + \varepsilon)$."

In another formulation, she stated verbally that, "only finitely many guys can be outside the room for you to have a party." The process of translation between the registers ended on the definition:

"* A sequence (x_n) converges to the real # L if given $\varepsilon > 0$, there exists $K \in N$ such that $\forall n \geq K,\ x_n \in (L - \varepsilon, L + \varepsilon)$."

During later lectures, the professor began to refer to these terms outside the party as "stragglers." Immediately after completing the definition of sequence convergence, the class worked to prove that $3n/_{2n+1}$ converges to 1.5, and they translated $3n/_{2n+1} \in (1.5-\varepsilon, 1.5+\varepsilon)$ into absolute value notation to say $\left|3n/_{2n+1} - 1.5\right| < \varepsilon$ in line with their work several weeks earlier on epsilon neighborhoods. Their proof, once completed, only justified the claim in terms of absolute value notation.

Thus the professor translated the intuitive register definition into the formal-symbolic register by replacing "stay as close to *L* as we wish" with epsilon neighborhood conditions and "going as far out as we wish" with index terminology. However, for the next several weeks of class meetings that covered the section on sequences, the professor continued to refer verbally to the party metaphor though everything written on the board was either graphical, intuitive register, or formal-symbolic register.

Linguistic Register	Sequence Elements	Proximity	Sequence order	"Cut off point"
Concrete metaphorical	People	Location	Time	Eventually
Informal mathematical	Terms	Closeness	Physical Arrangement	"Far enough out in the sequence"
Formal-symbolic	a_n	ε	Indices (n)	K

Table 1. Sequence convergence in the various registers.

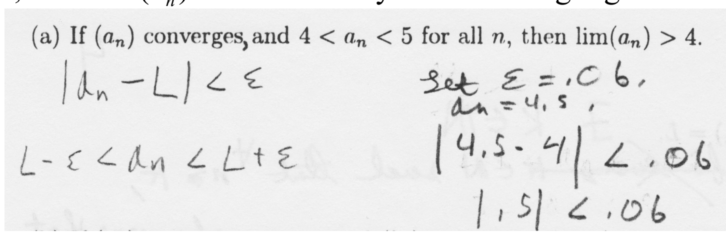
Tidus emerged as an instructive case of reasoning about sequence limits because of the central role the party metaphor played in his reasoning about limits. Over the course of talking with Tidus during interviews, several key aspects of his concept image of sequence convergence repeatedly appeared. When possible, each primary element of his concept image will be connected with tasks he attempted during exams or interviews to understand what forms of reasoning his concept image and concept definition supported and failed to support. Since all of the four interviews and two tests that dealt with aspects of sequence convergence fell within a calendar month, any quotes will be marked with their source type and day of the month: I-2, T-8, I-17, I-23, I-28, or T-29.

Neighborhood as place

Tidus consistently referred to the epsilon neighborhood as a place or a set rather than a measure of closeness between sequence terms and the limit value. He expressed his understanding of the definition of sequence convergence saying, "at some point … eventually all of those numbers will be in four's neighborhood" (I-2). He articulated that same definition later saying, "For any epsilon that you pick, an infinite amount of terms will be in that epsilon neighborhood and a finite amount of terms will be outside" (I-17). This appeared on the first test (T-8) in that six different times he juxtaposed absolute value and set-type expression such as $|x_n - 5| < .01/_2$ alongside $x_n \in \left(5 - .01/_2, 5 + .01/_2\right)$ or $|a_n - L| < \varepsilon$ alongside $L - \varepsilon < a_n < L + \varepsilon$. Consistently thereafter when discussing sequence convergence, Tidus would talk about numbers or terms being "in the epsilon neighborhood" rather than using closeness or approximation language. The only exception to this in the context of sequence convergence appeared on the first exam when

Tidus argued that $(a_n) \to 0$ and (b_n) converges $\Rightarrow (a_n b_n) \to 0$ by noting that "a_n gets closer and closer to zero."

This physical location view on epsilon neighborhoods appeared to limit Tidus' ability to reason about the artibrarity of epsilon or to change his value of epsilon. In the same way that the size of a house in the party metaphor cannot change, Tidus rarely referred to multiple values of epsilon in the same problem early on. This viewpoint led him to assert that "If (a_n) converges and $4 < a_n < 5$ for all n, then $\lim(a_n) > 4$" is false by the following argument:


(a) If (a_n) converges, and $4 < a_n < 5$ for all n, then $\lim(a_n) > 4$.

$|a_n - L| < \varepsilon$ set $\varepsilon = .06$, $a_n = 4.5$,

$L - \varepsilon < a_n < L + \varepsilon$ $|4.5 - 4| < .06$

$|.5| < .06$

Figure 2. Tidus' limit argument (T-8).

Tidus concluded "$\lim(a_n) > 4$ and a_n is trapped when $a_n = 4.5$ and $L = 4$" (T-8). Though his example and argument do not match the formal definition, it displays how his understanding of convergence at this point involved a single value for epsilon and "trapping" all of the values of the sequence in that epsilon neighborhood.

Tidus produced a mostly accurate proof that if a sequence converges to 5, then it cannot converge to 5.01. His chose $\varepsilon = .01/2$ and pointed out that the epsilon neighborhoods were disjoint. Since it is impossible for the terms of the sequence to be in the two neighborhoods at the same time, this produced a contradiction (T-8). Thus, Tidus' image of the epsilon neighborhood as a place supported success with certain types of reasoning, and success on tasks in which the fact that numbers are either inside or outside of the neighborhood proved important.

"Finding the K" as convergence

For some time, Tidus did not seem to reason at all about changing the value of epsilon relative to the same limit, but literally would "pick any" and leave it static thereafter. He understood that he could pick a convenient epsilon and did so on several tasks on the first test. During the second interview (I-17), he said:

- "If you show that the [sequence] can't go out [of the epsilon neighborhood], then it's going to converge."
- "It has to converge to an L in any epsilon neighborhood."
- "There should be one number in the [sequence] where the [sequence] converges to that point, where all of the terms are in that epsilon neighborhood."

Though Tidus does mention that the condition must hold "for any epsilon neighborhood," he also speaks in the first and third quotes as if convergence to the limit is not a global property of the sequence, but it is something that the sequence is "going to do" relative to the "one number in the [sequence] where the [sequence] converges to that point." Similarly during the third interview (I-23) he said, "We are finding a K_1 where it is going to converge to 5." Thus the focus of Tidus' view of convergence was on the bounding of infinitely many terms after K rather than upon the size of epsilon or the universal quantifier associated.

This notion supported his reasoning on some types of tasks. On the first test (T-8), the professor included a true-false question that stated: "Suppose the sequence X converges to -1. For $K \in N$, let X^K denote the sequence obtained by changing the first K terms of X to the value K

and leaving the rest of the terms as they are. Then, for all $K \in N$, the sequence X^K converges to -1." The professor spent some time during class articulating the point that a finite number of terms have no effect on the convergence of a sequence. Tidus' image of sequence convergence that focused on the existence of a point after which the terms must stay in the neighborhood allowed him to dismiss the changing of K terms as irrelevant for convergence. After writing out an example sequence X and X^3, he argued that the statement is true because, "If you change the value of the stragglers and did not the value of the terms [in] the ε-neighborhood, the sequence will still converge to L" (T-8). He used party metaphor language to refer to the finitely-many terms outside the neighborhood, and possibly this supported his understanding that their absence did not affect "the party."

The emergence of the arbitrariness of epsilon

During the third interview, Tidus began to juxtapose his use of epsilon language with "closeness" language in the context of the definition of Cauchy sequences. In the course of arguing why Cauchy sequences must be bounded, he said, "Well if they are getting close to each other, at some point they are going to be in each other's, I guess you could say epsilon neighborhood? Something like that because if they are getting closer to each other they can't be going out anywhere else" (I-23). Tidus continued to use his informal domain expression "at some point," and "in each other's… epsilon neighborhood," but he used epsilon as a means of expressing proximity as well.

Toward the end of the third interview, he wrote down a form of the formal-symbolic definition that began, "Let $\varepsilon > 0$…" and then appended to the end of his explanation, "and it should work for every epsilon" (I-23). This was the first time that Tidus has specifically referenced the generality or arbitrariness of epsilon in conjunction with the formal definition. When probed about the difference between "Given $\varepsilon > 0$…" and "For every $\varepsilon > 0$…", Tidus said "I don't see how it could be different… 'Cause either way you are going to have to pick… an epsilon. For every you are saying you are taking any epsilon, that K is going to be different for every epsilon… For a different epsilon there will exist a different K" (I-23). This was also the first time during interviews that Tidus expressed any awareness of this "dependence" between the quantities.

During Tidus' final interview regarding sequences (I-28), he worked on review problems for his upcoming exam and considered the question,

" (a) If (x_n) is a sequence and x_K is a term of the sequence such that $|x_n - x_K| < 0.1$ for all $n \geq K$, then which of the following, if any, are correct?

i) (x_n) is bounded
ii) (x_n) is Cauchy
iii) (x_n) has a Cauchy subsequence

(b) Let (x_n) be a sequence such that for every $\varepsilon > 0$, $|x_n - x_m| < \varepsilon$ for all $m,n \in N$. Then which of the following, if any, are correct?

i) (x_n) is Cauchy
ii) (x_n) is a constant sequence"

Tidus originally indicated that the sequence in part (a) must be convergent and thus Cauchy, but then caught himself and said, "Oh, but it's not for every epsilon… there could be an epsilon where it doesn't work" (I-28). About the latter question, he noted, "It's not picking a certain term where this happens… that could only happen if it's a constant sequence… cause it doesn't matter what term in the constant sequence you pick for it to happen, it's going to happen everywhere" (I-28). When pressed to prove why every term of the second sequence must be equal, Tidus could not produce the proper argument, but his image of what must happen if there was not a "certain term" led him to believe the sequence must be constant. He argued, "because here it doesn't say… that you have to pick a certain term in the sequence where this happens… [which] like the convergent ones [allows] stragglers." In both cases, the tasks required Tidus to reason about the definition of convergence or the definition of Cauchy in terms of multiple values of epsilon at the same time, and he was successful in reaching accurate conclusions.

Discussion

Tidus' initial reasoning about limit in which he never considered multiple values for epsilon showed the central place that the party metaphor took in his concept image of sequence convergence. His argument that the constant sequence $a_n = 4.5$ converges to 4 when $\varepsilon = .6$ conflicts with his calculus-based knowledge of sequence convergence. Tidus readily displayed in other contexts that he knew constant sequences converged to the constant value. When he made this argument, he reasoned about sequence convergence in terms of his concept image that was dominated by the party metaphor. For a neighborhood of size 0.6, all of the terms a_n are in 4's neighborhood, to use his parlance.

Tidus' reasoned and spoke about epsilon neighborhoods as places rather than treating epsilon as a measure of proximity. This further supports the assertion that Tidus' evoked concept image was dominated by properties of concrete space and physical location rather than properties of the number line such as quantity and numerical difference. Thus, when Tidus used terminology from either the concrete metaphorical or mathematical registers, these terms were most directly referencing elements of his concept image, and particularly the party metaphorical image of sequence convergence. This is true even for his argument on the first exam where he makes no specific reference to the party metaphor, but he forms an argument that directly conflicts with his computational knowledge of sequence convergence.

The professor articulated the conditions of sequence convergence in different linguistic registers; in particular, she described the interrelationship between the parameters ε, K, and n in the metaphor of a party and in the mathematical realm of sequences, neighborhoods, and indices. Because Tidus's schema for people attending a party was more accessible and well-developed than his mathematical schema for interpreting logical dependencies, he instantiated his concept of the definition of sequence convergence using his experiential schema of people and buildings. He was able to quickly and easily establish the interdependencies between a given epsilon, the cut-off point K, and the people at the party as represented by n. However, because the party scheme did not easily assimilate the universal quantifier on the size of the building (epsilon), it was not adopted into his conception of sequence convergence. Hence, his intermediate definition reflected this set of interrelationships between the parameter quantities: "at some point… eventually all of those numbers will be in four's neighborhood" (I-2).

Tidus received affirming feedback upon this definition schema because it properly facilitated his reasoning on certain tasks such as proving that the limit of a sequence is unique or that changing finitely many terms of a convergent sequence does not affect convergence. As he used this schema to solve new tasks and assimilate further experiences with sequences, the interrelationships between a given epsilon, the found K, and all n beyond K became encapsulated into a single concept in Tidus' thinking. This represented itself in his compact reference to the entire condition (his party metaphor-based ε-K-n schema) as convergence when he said, "We are finding a K_1 where *it is going to converge* to 5" (I-23). It was during this third interview, once Tidus displayed evidence of having encapsulated his "finding the *K*" such that… condition, that he also showed signs of assimilating the universal quantifier onto epsilon within this schema. I shall refer to his encapsulated schema as Tidus' *K*-condition. After writing the formal concept definition, which he previously reasoned about using his *K*-condition, he added, "and *it should work* for every epsilon" (I-23).

It is not evident from the interview data what exactly caused the disequilibrium that guided Tidus to begin this process of assimilation. He admitted during the second interview (I-17) that even though he could reproduce proofs that a given sequence converges to a real number *L*, he did not understand why the proof satisfied the definition (which he stated in terms of his *K*-condition). He lost points on his exam for the way in which he wrote the formal definition (T-8). He began to assimilate new ideas into his ε schema when he used it to describe the proximity of sequence terms in the Cauchy definition. He thus experienced both internal and external feedback that would have made him reflect upon the complete viability of his *K*-condition as the definition of sequence convergence.

Tidus was now able to reason more fluidly about the complete concept definition because it had been reduced to, "For every $\varepsilon > 0$, the *K*-condition holds." Encapsulating the *K*-condition allowed Tidus to reason more freely without having to actively coordinate the roles and quantifiers attached to *K* and *n*. By the final interview, Tidus displayed reasoning about sequence convergence that was compatible with standard interpretations and uses of that definition. He leveraged his *K*-condition schema to reason about a novel definition that lacked a *K* value. Tidus still used terminology that arose from the party metaphor when he identified that the absence of *K* doesn't allow any "stragglers." However, his reasoning did not display the limitations of the party metaphor because he reasoned using multiple possible values of epsilon. In this way, he had now accommodated the *K*-condition into a broader mathematical schema of the sequence convergence definition. This schema that organized the roles and quantifiers on the parameters ε, *K*, and *n* was now viable for assimilating novel definitions presented in the formal-symbolic register. The use of the term "stragglers" may even reflect the lack of a comparable term in the mathematical register; the phrase "tail of the sequence" is often used for the set of sequence terms after the K^{th} term, but no standardized term refers to those terms prior to the K^{th} term. In this way, Tidus' schema that had accommodated the *K*-condition from the party metaphor contained more tools for reasoning and communication than the formal statement of the definition entailed on its own.

Tidus' transition from the *K*-condition to reasoning in terms of the standardized definition stands in contrast to Alcock and Simpson's (2002) notion of inverting the property-category relationship in the sense that he did not begin by reasoning with examples or sets of examples, but rather developed a concrete mental model in which to conceptualize the logical conditions of sequence convergence. Tidus shifted the properties that for him characterized sequence convergence by abstracting from the concrete domain to the mathematical domain in which multiple values of epsilon can (and often must) be considered. This pathway toward the formal definition of limit did not begin with sets of examples, graphical representations, or the formal definition, but rather arose from a concrete metaphorical domain. Though Tidus could reproduce the statement of the concept definition of sequence convergence from memory, he interpreted that statement in terms of the *K*-condition couched in the party metaphorical domain until the point at which he encapsulated the *K*-condition and was able to accommodate the universal quantifier for epsilon.

Though it introduced some false lines of reasoning, the party metaphor proved formative for Tidus' reasoning because it allowed him to develop a property-based conception of convergence rather than a computation-based or exemplar-based conception. Tidus' intermediate conceptions of sequence convergence supported some lines of reasoning well but not others. By the end of the period of instruction, Tidus displayed the ability to accurately interpret novel conditions on sequence behavior stated in the formal-symbolic register. However, his reasoning still reflected useful aspects from earlier stages of his concept development. Thus the party metaphor appeared a non-trivial stepping-stone in Tidus' movement toward a concept definition and concept image of sequence convergence compatible with standard interpretations and toward fluency in the formal-symbolic register.

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